VOLUME LIII NUMBER ONE

The Mathematics Teacher

JANUARY 1960

Teaching load and qualifications of mathematics teachers
KENNETH E. BROWN

Solving simultaneous linear equations
ARNOLD WENDT

No space geometry in the space age!
CHARLES H. SMILEY *and* DAVID K. PETERSEN

Some heretical thoughts from an orthodox teacher
WALLACE MANHEIMER

Preparation in mathematics of mathematics teachers
JOHN J. KINSELLA

The official journal of
THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

The Mathematics Teacher *is the official journal of The National Council of Teachers of Mathematics devoted to the interests of mathematics teachers in the Junior High Schools, Senior High Schools, Junior Colleges, and Teacher Education Colleges.*

Editor and Chairman of the Editorial Board
ROBERT E. PINGRY, *University of Illinois, Urbana, Illinois*

Editorial Board
JACKSON B. ADKINS, *Phillips Exeter Academy, Exeter, New Hampshire*
MILDRED KEIFFER, *Cincinnati Public Schools, Cincinnati, Ohio*
DANIEL B. LLOYD, *District of Columbia Teachers College, Washington 9, D. C.*
E. L. LOFLIN, *Southwestern Louisiana Institute, Lafayette, Louisiana*
ERNEST RANUCCI, *State Teachers College, Union, New Jersey*

All editorial correspondence, including books for review, should be addressed to the Editor.

All other correspondence should be addressed to

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N. W., Washington 6, D. C.

Officers for 1959–60 and year term expires

President
HAROLD P. FAWCETT, *Ohio State University, Columbus, Ohio, 1960*

Past-President
HOWARD F. FEHR, *Teachers College, Columbia University, New York, New York, 1960*

Vice-Presidents
IDA BERNHARD PUETT, *Atlanta, Georgia, 1960*
E. GLENADINE GIBB, *Iowa State Teachers College, Cedar Falls, Iowa, 1960*
MILDRED B. COLE, *Waldo Junior High School, Aurora, Illinois, 1962*
PHILLIP S. JONES, *University of Michigan, Ann Arbor, Michigan, 1962*

Executive Secretary
M. H. AHRENDT, *1201 Sixteenth Street, N. W., Washington 6, D. C.*

Board of Directors
CLIFFORD BELL, *University of California, Los Angeles 24, California, 1960*
ROBERT E. K. ROURKE, *Kent School, Kent, Connecticut, 1960*
ANNIE JOHN WILLIAMS, *State Department of Public Instruction, Raleigh, North Carolina, 1960*
FRANK B. ALLEN, *Lyons Township High School, La Grange, Illinois, 1961*
BURTON W. JONES, *University of Colorado, Boulder, Colorado, 1961*
BRUCE E. MESERVE, *Montclair State College, Upper Montclair, New Jersey, 1961*
PHILIP PEAK, *Indiana University, Bloomington, Indiana, 1962*
OSCAR F. SCHAAF, *University of Oregon, Eugene, Oregon, 1962*
HENRY VAN ENGEN, *University of Wisconsin, Madison, Wisconsin, 1962*

Printed at Menasha, Wisconsin, U.S.A. Entered as second-class matter at the post office at Menasha, Wisconsin. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in paragraph 4, section 412, P. L. & R., authorized March 1, 1930. Printed in U.S.A.

The Mathematics Teacher

volume LIII, *number* 1 January 1960

Teaching load and qualifications of mathematics teachers, KENNETH E. BROWN 2

Solving simultaneous linear equations, ARNOLD WENDT 12

No space geometry in the space age? CHARLES H. SMILEY and DAVID K. PETERSEN 18

Some heretical thoughts from an orthodox teacher, WALLACE MANHEIMER 22

Preparation in mathematics of mathematics teachers, JOHN J. KINSELLA 27

DEPARTMENTS

Historically speaking,—HOWARD EVES 33

Reviews and evaluations, KENNETH B. HENDERSON 39

Tips for beginners, JOSEPH N. PAYNE *and* WILLIAM C. LOWRY 42

Have you read? 26, 32; *What's new?* 21

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Report of the Nominating Committee 46

Your professional dates 53

Program for joint meeting of MAA and NCTM 54

National Science Foundation Academic Year Institutes 55

THE MATHEMATICS TEACHER is published monthly eight times a year, October through May. The individual subscription price of $5.00 ($1.50 to students) includes membership in the Council. For an additional $3.00 ($1.00 to students) the member may also receive *The Arithmetic Teacher*. Institutional subscription: $7.00 per year. Single copies: 85 cents each. Remittance should be made payable to *The National Council of Teachers of Mathematics*, 1201 Sixteenth Street, N. W., Washington 6, D. C. Add 25 cents for mailing to Canada, 50 cents for mailing to foreign countries.

Teaching load and qualifications of mathematics teachers[1]

KENNETH E. BROWN, *U. S. Office of Education, Washington, D. C.*

Are mathematics teachers generally qualified to face the decisions of curriculum revision they must make? Do they have the time and facilities needed to make wise changes?

IN RECENT years there have appeared in the press many articles critical of the public schools. The President of the United States has asked citizens to scrutinize the schools to see if they meet the stern demands of the era we are entering.

One cannot examine the schools intelligently without considering the teachers and the pupils. They are the very heart of our school system.

THE NUMBER OF HIGH SCHOOL PUPILS TAKING MATHEMATICS

In the fall of 1956 the public school teachers were teaching nearly 4.4 million mathematics students in grades 9–12. This was nearly a 50 per cent increase since 1948–49. Of course, this number includes many who were taking non-college preparatory mathematics. As a frame of reference from which to appraise the number 4.4 million, perhaps we should recall that this number is more than 21 times the number of high school pupils in 1890. That is, the number enrolled in mathematics in grades 9–12 in 1956 was approximately 21 times the total high school enrollment in 1890. Approximately half of the 4.4 million pupils are enrolled in college-preparatory mathematics.

The number enrolled in intermediate algebra, a college-preparatory course, for example, was about one-third of the eleventh-graders. Therefore, we can probably assume that about a third of our graduates have had three years at least of college preparatory mathematics. The number taking solid geometry and trigonometry, normally a twelfth-year subject, was about 12½ per cent of the students in the twelfth grade; thus, we might assume that approximately one-eighth of our high school graduates have had four years or more of mathematics. These assumptions are confirmed by the NEA in a study of enrollments and offerings in mathematics and science, reported in the NEA Research Bulletin, Volume 36, No. 3, October, 1958, which I shall refer to simply as the NEA study.

The NEA study reports that in 1957, 460,000 high school graduates took three years of mathematics. This is approximately one-third of all graduates. The study reports that 14.4 per cent of the 1957 high school graduates had completed four years of mathematics. Also, the study reports that 6,235,000 high school youths were taking mathematics in 1957. However, it should be observed that this number includes those students who are taking mathematics in the seventh and eighth grades. Since there are approximately 2,000,000 students in the seventh- and eighth-grade secondary schools, this would leave approximately 4,235,000 taking mathematics in grades 9–12, which is comparable to the 4,400,000 indicated in 1956

[1] Speech given by Dr. Brown at the annual meeting of the National Council of Teachers of Mathematics in Dallas, Texas, April, 1959.

by the Office of Education survey. The NEA study shows that about 175,000 students who had completed four years of mathematics graduated from the secondary schools in 1957. The Office of Education survey in 1956 showed that between 160,000 and 200,000 students were taking solid geometry and trigonometry, normally twelfth-year subjects. Thus these studies indicate that at the present time there are probably about 4½ million students in grades 9–12 in the United States who are enrolled in high school mathematics. However, many of these students are not in the normal college-preparatory type of mathematics. These surveys seem to show that approximately one-third of all the graduates from a secondary school have had three years of mathematics, and perhaps one-eighth or slightly more have had four years of mathematics.

Of course the question immediately comes up: Should more students be taking mathematics? There seem to be indications that there is a shortage of students studying mathematics and science.

Frontier thinkers in mathematics agree that in the future a knowledge of mathematics will be indispensable to many of our citizens. Dr. James B. Conant, in his report[2] on the study of the comprehensive high school, recommends that the top 15 per cent of our high school population should have four years of mathematics. But at the moment no national goals have been established to indicate the exact number that should be taking mathematics.

Although the number who are taking mathematics or who should be taking mathematics is not clear, two things are evident. First, many of the academically talented pupils do not develop their talents in mathematics. They do not choose to take four years of high school mathematics. Also, some pupils who are in more advanced high school mathematics courses lack the ability to profitably pursue the course. I am sure that you can testify to the fact that many pupils in your classes should have had better guidance.

The second thing that is quite clear is that all students do not have an opportunity to develop their talents. For example, in 1956 about one-fifth of the schools did not offer plane geometry. Although these were small schools with an average tenth-grade enrollment of only 35 pupils, they included 6½ per cent of all the pupils in the United States secondary schools. As the pupil in a small school continues his work in mathematics, the possibility of selecting an advanced course becomes less and less. Twenty-eight and one-tenth per cent of all the twelfth-grade pupils were in a school that offered neither trigonometry nor solid geometry. It is not in harmony with the spirit of a democracy to deny a student proper education because he is in a small school.

As we have observed, 175,000 pupils approximately have completed four years of mathematics. However, in the small schools there were 100,000 seniors who couldn't obtain advanced mathematics. The NEA has emphasized this condition by stating, "Large schools have 1.25 times the enrollment of small schools, but they graduate 2.1 times as many pupils with four years or more of mathematics." The NEA study continues, "It becomes obvious that larger schools are affording their students opportunities for extra academic miles and that students are taking advantage of them."

If the pupils in the small high schools were provided with advanced mathematics courses, that is, trigonometry, solid geometry, and a twelfth-grade course, and if they took the courses in proportion to the pupils in the large schools, then we would increase the number of pupils with four years of mathematics by more than 16 per cent. If equal opportunities are to be provided for the development of all youth, more mathematics must be offered in small schools.

[2] James B. Conant, *The American High School Today* (New York: McGraw-Hill Book Company, 1959).

Perhaps it was this conviction that led the National Association of Secondary School Principals to state, "In the national interest it is desirable to offer advanced study in mathematics to capable and interested students regardless of the smallness of the class." I shall quote further from the Association's report in the September, 1958, *Bulletin*, "It is recommended that advanced courses in science and mathematics be made available without regard to smallness of the class enrollment. At very small schools or where it is impossible to provide advanced instruction to two or three students the use of correspondence courses, resource persons in the district, independent supervised study and other approaches can be employed."

Dr. Conant recommends in *The American High School Today* that small high schools be consolidated so that no high school has a graduating class of less than 100 pupils. Consolidation will help to make possible a school with large classes of pupils in advanced mathematics. However, we have only 4,400 high schools with enrollments of 500 and above. Consolidation of 19,000 schools will require several years and the construction of many new buildings. In the meantime, provision should be made to develop all talent, even if it is expensive to develop in small schools.

Occasionally one will hear the "good old days" described in glowing terms. However, in 1890 the total number of high school mathematics pupils was 135,000 compared to 4.4 million in 1956. In 1956 the trigonometry enrollment alone was as great as the entire high school enrollment in 1890. Our high school mathematics enrollment is thirty-two times the mathematics enrollment in 1890.

Perhaps the status of enrollments in mathematics can be summarized this way. Our schools are teaching more students mathematics than at any time in the nation's history. At the present time 12 to 15 per cent of our high school graduates have four years of mathematics, yet evidence indicates more students should develop their talents in these areas. However there has been no national study or agreement to indicate the percentage that should have four years of mathematics. Many students do not have an opportunity to develop their talent in mathematics because advanced courses in mathematics are not offered in many school systems. Courses can be provided, if we wish to bear the expense, by consolidating small schools or by instruction to small classes.

The nearly 5 million mathematics pupils in our public and nonpublic high schools are not divided evenly among the teachers. Although the average algebra class has 30 pupils, the average class size does not really reflect the classroom situation. There is no average class. All classes are either larger or smaller than the average. One survey[3] showed that 60 per cent of the mathematics classes of the public junior high schools contained more than 30 pupils. An average of 30 indicates many large classes where provision for individual differences would be very difficult if not impossible. These large classes are usually in the ninth grade. At the very time when pupils need individual classroom assistance to understand basic mathematics they unfortunately find themselves in the largest classes.

In considering the teaching load one should not confine the discussion to the 5 million mathematics pupils in the public and nonpublic high schools. Many teachers teach other subjects. A survey made in 1957–58 showed that nearly half the mathematics teachers—even those in high schools with enrollments of 500 or more—did not confine their teaching to mathematics exclusively. About 19 per cent of the teachers, in addition to teaching mathematics, also taught science: 13 per cent taught physical education; 7 per cent

[3] Kenneth E. Brown, *Mathematics in Public High Schools*, Bulletin 1953, No. 5 (Washington, D. C.: U. S. Government Printing Office, 1953), p. 37.

taught social studies; 5 per cent taught English.

In addition to the various teaching assignments, 38 per cent of the teachers had one or more study halls, 14 per cent devoted one or more periods to guidance, 12 per cent had administrative duties, 6 per cent had one or more periods of committee work. Most of the teachers received no help. The per cent of teachers who had noncertified teachers to assist them was negligible. Approximately 7 per cent had student teachers. However, most teachers will agree that, although student teachers may assist on routine duties, in general they require many hours for consultation. The mathematics teacher cannot expect much consultative help in the area of specialization from his superior, even if one is available. A survey showed that three-fourths of the teachers had access only to general supervisors.

The extracurricular activities of the mathematics teacher are an additional burden. In small high schools, teachers report that they devote four hours or more to extracurricular duties. In large high schools one-half of the teachers devote four hours or more to these out-of-class activities. The teachers in the smaller schools devote more time to extracurricular activities but less time to mathematics clubs and mathematics contests than the teachers in large schools.

Preparation of curricular materials is an additional duty

Another duty of the mathematics teacher is the preparation of the course of study or teaching guide. Is there a mathematics teacher who has not heard the principal at the first staff meeting of the new school year say, "This year the curriculum emphasis will be on mathematics. A mathematics teacher will prepare a course of study"?

In 1952 the Office of Education published a bulletin entitled, "Curriculum Materials in High School Mathematics." In this bulletin approximately 135 teaching guides or courses of study that were in current use were analyzed. Several of these guides or courses of study were rather old but were still being used. Recent surveys indicate that the picture of the curriculum revision has changed substantially since 1952.

A questionnaire to a randomly selected sample of high schools in 1958 indicated that 38 per cent of the schools have a course of study or teaching guide. About 27 per cent of the schools reported that one or more teachers had taken part in 1958 in developing or revising the local teaching guide. Also, about one-third of the schools said that they plan to develop a local teaching guide or course of study during the present year. Last year the NEA research department found that 43 per cent of the schools indicated that at some time during the past two years they had a mathematics curriculum project in progress. The large high schools were even more active than the smaller ones; nearly 70 per cent of the large high schools were engaged in such activity.

The question naturally arises as to why this emphasis on curriculum revision. An Office of Education study indicates that some outside state or local pressure has been put on the schools to increase and bolster up mathematics instruction. About 40 per cent of the schools in the sampling study indicate that they have had some pressure. About 17 per cent say that there has been little or no pressure.

Has this pressure had any effect on changing mathematics instruction programs? About one-third of the principals indicate that some change has resulted. About 10 per cent say that there has been considerable change; about one-fourth indicate no change. It is perhaps safe to conclude that, although some of the curriculum revision began before Sputnik, much of the revision is an attempt to provide better mathematics instruction in the national interest. In any case, curriculum revision has been taking place.

In spite of the many duties of a mathe-

matics teacher, do teachers try to improve their professional competency? The answer is yes. According to a survey 47 per cent of the teachers do professional reading. In fact, one-third of the teachers in large high schools report that they are regular readers of THE MATHEMATICS TEACHER. Another 20 per cent attend workshops or college classes. The attendance at the National Science Foundation Institutes and in summer courses testifies to the desire of teachers to improve their professional talents. A survey shows that, in the opinion of both teachers and principals, the workshops and institutes are very helpful to mathematics teachers. In this same survey undergraduate courses in mathematics were rated very helpful by 68 per cent of the teachers. However, only 29 per cent of the teachers rated the education courses very helpful. Fifty-nine per cent of the teachers indicated that they thought more courses in college mathematics would be helpful to them. But only 15 per cent thought the same about courses in education.

In discussing education these days it seems to be the fad to refer to U.S.S.R. education. This discussion will be no exception. It seems that the little "peep peep" from outer space really frightened some people. Perhaps a little fear is a good thing. However, have you ever noticed that, when we suddenly think we are weak and insecure, we have a tendency to look for a scapegoat? Many times that scapegoat is one of our stable institutions. If the morals of the G.I.'s are questioned, the church may receive the blame. If the U.S.S.R. outwits us by putting up a dying dog in a flying box before we do, the mathematics teachers are to blame.

I just do not concur with such reasoning. It is the reasoning of a child. My little boy builds his house of blocks. As he places the last block the house goes tumbling down. He looks to his mother, who is on the other side of the room, and says, "See what you made me do?" If schools fail in their mission, it is the fault of every citizen. We get the schools we want and are willing to pay for. We have good schools and the contributions of our scientists testify as to their excellency. Before I am quoted out of context, I must hasten to add that I agree with each one of you that mathematics teachers want to do a better job and that they are trying to improve their competency.

U.S.A. vs. U.S.S.R. EDUCATION IN MATHEMATICS AND SCIENCE

Let us look at a few facts and figures.

U.S.A.

In 1956 about 1,300,000 graduated from the 12th grade.

In 1956 78% of the appropriate age group, 14–17, were in high school.

About 60% of the 17-year-olds graduate from high school.

It is estimated that in 1961 45 million pupils will be enrolled in grades K-12.

U.S.S.R.

In 1956 about 1,300,000 graduated from the ten-year-school system.

In 1956 only 33% of a comparable age group were in secondary school.

Only about 5%–10% graduate from secondary school.

Estimates are that by 1961 65 million pupils will be enrolled in grades 1–10.

An examination and analysis by ETS at Princeton indicates that the mathematics examination in the U.S.S.R. ten-year-school and the U.S.A. college board examination are similar in depth and over substantially the same material. The U.S.S.R. textbooks do not contain much so-called modern mathematics and their presentation is certainly very formal and traditional. In most classes, the U.S.S.R. high school mathematics textbooks are reprints of books whose contents have remained the same for the last twenty years.

It was my privilege to attend an international conference on mathematics education in which a Soviet delegation took part. After a week or more of discussion of problems in mathematics education, I concluded that their methods are inferior to ours and in only a few instances did they have mathematics content on the high school level that might be profitable for us to adopt. What do these facts mean to me? They mean that in the United States the students who are taking four years of mathematics are getting as sound a preparation in the subject as students in the U.S.S.R. However, more pupils in the U.S.S.R. than in the U.S. are taking such training. Should more students take mathematics in this country? That question should be answered by all citizens. National goals should be determined by the nation. This is not the responsibility of the mathematics teacher alone.

Qualifications of mathematics teachers

In a discussion of mathematics education, this question comes up, "What are the mathematics teachers' qualifications?"

From some of the comments one hears it might be concluded that teachers are just a bunch of old maids drifting from school to school looking for a husband. However, this is not the case. A questionnaire sent to five per cent of the schools in the United States indicates that two-thirds of the teachers are men and three-fourths of the teachers are married. That is a pretty good cross section of our population. The survey also indicates that 81 per cent of the teachers who were teaching last year had taught in that same school the year before. Since 12.7 per cent of the mathematics teachers had not taught the year before, the remaining percentage (6 per cent) had changed jobs. Of course, many of the mathematics teachers are young; 30 per cent had taught three years or less. We should do everything in our power to encourage the capable young teachers to stay in their positions.

How well qualified are the teachers? According to the NEA survey, new teachers coming into the profession are not as well prepared as the present staff. Last year the Office of Education made a survey of the qualifications and teaching load of mathematics teachers in three selected states, which I shall call the three-state study. In this study a mathematics teacher was defined as anyone who taught one or more classes in mathematics. The qualifications of the teachers were secured by a study of their transcripts and also from a questionnaire sent directly to the teacher.

This study indicated that the mathematics teacher on the average had 23 hours in mathematics. It should be pointed out that, in many cases, much of this mathematics was below the calculus. Nearly one-third of the teachers reported an undergraduate major in mathematics. The NEA study showed a similar situation in 1957. Research indicated that 40.9 per cent of the teachers had 30 hours in mathematics.

In contrast to those who are well-prepared in mathematics, the three-state study showed 7.1 per cent of the teachers with no collegiate mathematics at all. The NEA study reported 8.7 per cent of teachers in small junior high schools to be without college mathematics. In the three-state study an average of 39 per cent of the teachers had not had the calculus. This percentage varied among the states from 16.2 to 56.5. Nearly one-third of the teachers had a master's degree, but in most cases it was not in mathematics. A national survey of a five per cent sample of teachers revealed that 54 per cent of them had taken no graduate courses in mathematics.

In contrast to 23 hours in mathematics for the average teacher was his 31.3 hours in education. In fact, about 50 per cent of the teachers had more than 25 hours in education in addition to practice-teaching and methods courses in the teaching of mathematics. In 1957 a survey of five per cent of mathematics teachers showed that

75 per cent of the teachers had an average grade in all mathematics courses of *B* or above. In education courses 80 per cent had an average grade of *B* or above. These studies seem to show that some teachers are well prepared in mathematics, but that, on the other hand, many teachers are lacking.

A distribution of the present mathematics teachers' bachelor's degrees by years shows a distribution in the center of the curve. Between 1950–57, 35 per cent of the bachelor's degrees were received; 30 per cent were received twenty-five or more years ago. Many of the master's degrees have been received recently. For example, nearly as many master's degrees were received since 1950 as all those received before 1950. The academic qualifications of the average teacher, like the average class size, does not indicate the real situation. A large group of the mathematics teachers are well prepared and another large group have a meager academic preparation.

In a discussion of mathematics education perhaps one should consider ways to help the present teachers and to improve the present situation in mathematics education.

What can be done to help teachers?

Perhaps one of the first things that can be done to help a teacher is to provide the proper grouping of pupils. It does not seem realistic to give a teacher 45–50 students with abilities from the second-grade to the eighth-grade level or from the third-grade to the twelfth-grade level. However, in many cases this is what is happening. The Office of Education survey status study previously referred to indicated that in about half of the high schools there is ability grouping of students in high school mathematics. One might assume that fewer of the schools with small classes group pupils than those with larger classes. But it was interesting to note that about the same percentage of the schools grouped students in algebra, where classes are usually large, as in trigonometry, where in general, classes are much, much smaller. Evidently, if there is a desire to group students, a way can be found to do so.

Many committee reports have indicated that there should be homogeneous grouping, but these same reports have pointed out that homogeneous grouping will not take care of all the defects of poor teaching. This month the President's Committee on Scientists and Engineers published a report on teaching of science and mathematics. The report says: "We believe that there should be homogeneous grouping of students in science and mathematics grades 7–12 and we are happy to observe this practice is gaining widespread favor." This procedure may be helpful to mathematics teachers, but mathematics teachers must be warned that in any kind of grouping provisions must be made for individual difference. In any kind of grouping there will be differences in the mathematics ability of the students. As someone has said, no matter how thin you cut the bologna there is always a top and a bottom.

Another way we may help the mathematics teacher is to guide the proper students into mathematics. This does not mean guidance should be based upon ability alone. The interest of the student, his goals, and his ambitions must be considered in guidance. At the present time we lack conclusive evidence as to the number or percentage that should be guided into advanced courses of mathematics. However, there seems to be evidence that more students could profit from the experience of taking mathematics than are doing so.

Another thing that would help mathematics teachers would be to reduce the number of class preparations they have to make. In the three-state study to which I referred, one-fourth of the teachers were teaching two classes or less in mathematics. This means they had other preparations to make. Mathematics was only incidental to the main teaching job. In fact only 45 per cent of the teachers were devoting full time to mathematics.

Many teachers had other duties in addition to teaching. A reduction of extracurricular activities would be very helpful for mathematics teachers. Of course, someone may say that a mathematics teacher should take care of mathematics contests and mathematics clubs. Perhaps this is true, but there are many other extracurricular activities which might be taken over by someone else who is less qualified in the field of mathematics. For example, in the status study referred to, principals of the high schools indicated that more of the mathematics teachers were devoting time to athletic contests than to mathematics contests. More of the teachers were devoting time to the supervision of grounds than to field trips in mathematics. In fact, twice as many teachers were devoting time to athletic contests as to mathematics clubs or science clubs. Approximately one-third of the teachers said they spent four or more hours per week on this type of extracurricular activity. Perhaps many of these duties should be assumed by other personnel.

Of course, equipment and facilities are needed by mathematics teachers. The 1957 report by the NEA indicated that the number of calculators, adding machines, or comptometers in schools was very small. In fact, the survey shows that only 90 per cent of the schools had protractors or compasses and only about three-fourths had models for geometric study. Less than 60 per cent even had graph boards. There was a large slide rule for demonstration in less than half of the schools. In a status survey, the Office of Education asked teachers to rate some 20 items of equipment and also to fill in other items they thought would be especially valuable to them. Mathematics models led the list. Teachers do need facilities and equipment; research has pointed out that they also need instruction in how to use them.

Teachers need more in-service education. And teachers are one of the first groups of people to admit that they need this type of help. In the status study 85 per cent of the mathematics teachers indicated that they would like to have more workshops. About one-third of the teachers reported that more advanced mathematics courses would be helpful to them. When teachers ranked 20 topics from 1 to 5 according to need, more than half placed the teaching of modern mathematics in first or second place. About one-third of the teachers thought that they could be helped by courses in the teaching of mathematics. Less than one-fourth felt that education courses would be helpful. When teachers were asked to rank the value of their undergraduate mathematics courses on a 5 point scale, only about 5 per cent indicated that their undergraduate mathematics courses were not helpful to them. However, for the education courses, the percentage was about 27. It is quite clear from these surveys that mathematics teachers feel that they need to know more about mathematics as well as better ways of presenting the material.

The high school principals seem to agree with the teachers. On a 5-point scale, the principals rated workshops or institutes at the top. More than 80 per cent of the principals indicated that the teachers that they had sent to workshops or institutes returned with a marked improvement in mathematics instruction. In fact, three-fourths of these principals said that the teachers who returned from the workshop had shared their experience with others and helped them to improve their instruction.

The teachers might be expected to want more courses in mathematics since 54 per cent of them have had few or no graduate courses in this area. The status survey showed this to be true. Since teachers want greater knowledge of the subjects, more in-service training programs such as workshops and institutes should be provided for them.

Of course, there is another change which must be made if mathematics education is to be improved rapidly. This last point is

certainly not the least. Salaries of mathematics teachers must be increased. A survey last year showed that about one-half of the mathematics teachers got a raise of $300 or more for the year. Another third got a raise that was less than $300. On an average, salaries are going up a little faster than the normal inflation.

Summary

Recent surveys seem to indicate that about 4½ million students are enrolled in mathematics grades 9–12. However, many of these students are not enrolled in college preparatory mathematics. Approximately one-third of the students will graduate with three years in mathematics, and perhaps 15 per cent with four years of mathematics. The number that should graduate has not been determined. Neither has the percentage of students that should be taking advanced mathematics been determined on a national basis. However, there seems to be evidence that more students should take mathematics than are doing so at the present time. The number could be increased by perhaps 15 to 20 per cent by providing the opportunity for all students to take mathematics. This would mean offering mathematics courses in the small high schools where some 100,000 students graduate annually without having had this subject.

On the one hand, the teachers of our high school mathematics students are in many cases well qualified. About one-third have majors in the area of mathematics. However, on the other hand, many of the teachers are not qualified. In fact, some have no collegiate training in mathematics at all. Their qualifications show that they have many more hours in education than in mathematics.

A survey of our high schools indicates that many persons are actively engaged in curriculum revision. Perhaps one-fourth of the teachers last year were engaged in developing local teaching guides or curriculum material. Nearly a third of the high school principals indicated that they had plans for developing local teaching guides and courses of study this year. Nearly 40 per cent of the principals indicated that there has been pressure to do something about improving mathematics instruction. In fact, about one-third of these principals indicate that there has been some change as a result of this pressure.

To improve instruction in mathematics, mathematics teachers need assistance. They need the children grouped according to ability, interest, and aptitude. Also they need the right students guided into mathematics courses. It is certainly undemocratic to frustrate the slow student by placing him in an advanced mathematics course for which he is not competent. It is as equally undemocratic to cheat the scholarly pupil out of his birthright by placing him in a general mathematics course. Proper guidance is imperative for maximum education of the youth.

Because we do have a shortage of mathematics teachers and because it is expensive to train a mathematics teacher, the mathematics teacher's time should be devoted to teaching students rather than to clerical work. The status study indicates that 10 per cent of our mathematics teachers have two or more study halls if they are teaching in the junior-senior high school or in a regular high school. This same study indicates that 55 per cent of mathematics teachers have less than five classes in mathematics. Many teachers are teaching several subjects, so that they have many preparations to make.

Mathematics teachers need to be relieved of certain extracurricular activities. It is an extra burden upon mathematics teachers and perhaps a waste of the mathematics teacher's time to look after study halls. The supervision of lunch rooms, school grounds, coaching of athletic contests, and similar activities might be done by someone who is not trained in an area in which there is a manpower shortage.

Mathematics teachers need in-service education. They have asked for in-service

education. Principals have indicated that workshops or institutes have been very valuable to mathematics teachers. Mathematics teachers have rated the need for more courses in mathematics as very important in their work. In-service education should be made possible for mathematics teachers.

In addition to the in-service training, teachers must have the proper equipment; they must have graph boards, models for display. And, of course, one of the most important conditions that must be met is concerned with salaries. The salaries of mathematics teachers must be commensurate with the job that they are doing.

Letters to the editor

Dear Editor:

Being interested in mathematics, I send you a greeting, expressed by math problems.

SERIES OF PROBLEMS

Every number can be expressed by single digits. For example:

$$1958=(111-11-11)\times(1+1)\times(11),$$

$$1958=(222\div 2-22)\times 22,$$

$$1959=(22\times 2)^2+22+\frac{2}{2}$$

$$=(19-59)-19\times 59-(19^2-59^2),$$

$$11,111=2,222\times\left(2^2+\frac{2}{2}\right)+\frac{2}{2}.$$

i. To express 1,000,000 by single digits 1's, 2's, 3's, · · · 9's, and all the digits, zeros as well, all mathematical operations can be indicated.

ii. To express 1,000,000 by all ten digits, but use them only once.

iii. To express suitable roots $\sqrt{\ }$ by simple infinite series of single digits. Perhaps somebody will meet with success in finding a solution of this problem for number π.

SOLUTIONS:

i. $\sqrt{(11-1)^{11+1}}$, $\left(\frac{22-2}{2}\right)^{2+2^2}$,

$\left(3\times 3+\frac{3}{3}\right)^{3+3}$, $\left(\frac{33-3}{3}\right)^3\times\left(\frac{33-3}{3}\right)^3$,

$(\sqrt{4}+4\sqrt{4})^{4+\sqrt{4}}$, $(5+5)^{5+5/5}$,

$\left(\frac{66-6}{6}\right)^6$, $\left(\frac{77-7}{7}\right)^{7-7/7}$,

$(8+\sqrt{\sqrt{8+8}})^{8-\sqrt{\sqrt{8+8}}}$, $\left(9+\frac{9}{9}\right)^{\sqrt{9}+\sqrt{9}}$,

0!000 000, for the factor $0!=1!=1$.

ii. $80\times 625\times[7+9+\sqrt{(1+3)\times 4}]$.

iii. IN THE CONTINUED FRACTIONS:

$$\frac{\sqrt{5}-1}{2}=\frac{1}{1}\,\frac{1}{+1}\,\frac{1}{+1}\,\frac{1}{+}\cdots=0.618\ 034\cdots$$

$$\sqrt{1^2+2}-1=\frac{2}{2}\,\frac{2}{+2}\,\frac{2}{+2}\,\frac{2}{+}\cdots=0.732\ 051\cdots$$

$$\sqrt{\left(\frac{3}{2}\right)^2+3}-\frac{3}{2}=\frac{3}{3}\,\frac{3}{+3}\,\frac{3}{+3}\,\frac{3}{+}\cdots=0.791\ 288\cdots$$

$$\sqrt{2^2+4}-2=\frac{4}{4}\,\frac{4}{+4}\,\frac{4}{+4}\,\frac{4}{+}\cdots=0.828\ 427\cdots$$

$$\sqrt{\left(\frac{5}{2}\right)^2+5}-\frac{5}{2}=\frac{5}{5}\,\frac{5}{+5}\,\frac{5}{+5}\,\frac{5}{+}\cdots=0.854\ 119\cdots$$

$$\sqrt{3^2+6}-3=\frac{6}{6}\,\frac{6}{+6}\,\frac{6}{+6}\,\frac{6}{+}\cdots=0.872\ 983\cdots$$

$$\sqrt{\left(\frac{7}{2}\right)^2+7}-\frac{7}{2}=\frac{7}{7}\,\frac{7}{+7}\,\frac{7}{+7}\,\frac{7}{+}\cdots=0.887\ 482\cdots$$

$$\sqrt{4^2+8}-4=\frac{8}{8}\,\frac{8}{+8}\,\frac{8}{+8}\,\frac{8}{+}\cdots=0.898\ 974\cdots$$

$$\sqrt{\left(\frac{9}{2}\right)^2+9}-\frac{9}{2}=\frac{9}{9}\,\frac{9}{+9}\,\frac{9}{+9}\,\frac{9}{+}\cdots=0.908\ 327\cdots$$

Or in the form of a compound division, derived from the continued fraction:

$$\cdots a\div\left\{a+\left\{a\div\left\{a+\left[a\div\left(a+\frac{a}{a}\right)\right]\right\}\right\}\right\}\cdots$$

where $a=1, 2, 3, \cdots, 9$.

You can make use of my letter.

Yours sincerely,
ADOLF KODYM
Růžová 17
Prague 3
Czechoslovakia, Europe

Solving simultaneous linear equations

ARNOLD WENDT, *Western Illinois University, Macomb, Illinois.*

A method of solving sets of linear equations, especially useful if there are more than three variables.

THE APRIL 1951 issue of THE MATHEMATICS TEACHER contains an article[1] by Professor C. C. MacDuffee on the solving of sets of simultaneous linear equations. We feel the systematic method of elimination discussed in this article should be introduced in high school and at the latest in the freshman college mathematics. Because most texts in current use do not mention this method, perhaps a further exposition may be of value.

This method should be introduced whenever simultaneous linear equations are treated. The first method used in most texts for equations in two and three unknowns is usually elimination, and this should not be difficult to extend to the method about to be discussed.

For only two and three unknowns this method offers no special advantages, but for more unknowns it is more economical and can be applied more generally than determinants. It is applicable to any system of linear, simultaneous equations. However, determinants may still be better to use in instances where the coefficients of the unknowns are functions involving variables rather than constants. The following method can also be readily adapted to use of desk calculators and automatic computing machines.

We shall first illustrate the method and then introduce refinements which will reduce the labor somewhat. Suppose we are given the two equations $2x+5y=-2$ and $3x-2y=-4$. We eliminate x from the second equation by multiplying the first equation by 3 and adding this result to minus two times the second equation. This leaves us with the following two equations:

$$2x+5y=-2,$$
$$19y=2.$$

Solving the second equation for y and substituting this value in the first equation results in the solutions $y=2/19$, $x=-24/19$. These answers should, of course, be checked in the given equations.

So far we have seen nothing different from the usual textbook presentation. We consider next the following three equations:

$$x+2y-3z=6,$$
$$2x+5y-2z=4,$$
$$x+3y-\ z=0.$$

We eliminate x from the second and third equations by adding -2 times the first equation to the second and adding -1 times the first to the third equation. (Fewer errors in arithmetic are made, we find, by avoiding "subtracting" one equation from another.) After performing the aforementioned operations we have the following set of equations:

$$x+2y-3z=6,$$
$$y+4z=-8,$$
$$y+2z=-6.$$

We now eliminate y from the third equation by adding -1 times the second to it, giving us these three equations:

$$x+2y-3z=6,$$
$$y+4z=-8,$$
$$-2z=2.$$

Solving the last equations for z and substituting in the second and first equations

[1] C. C. MacDuffee, "Linear Equations Without Determinants," THE MATHEMATICS TEACHER, XLIV (April 1951), pp. 233–234.

respectively gives us the solution $z=-1$, $y=-4$, and $x=11$.

The careful reader may have noted an omission in the discussion. The last set of equations above is not identical with the given set. Yet the solutions of the final set are in fact the solutions of the given set. The following theorem justifies the foregoing procedure by showing that sets of equations obtained from one another by the method used are equivalent in the sense that their sets of solutions are identical. We present the theorem in a more general form than necessary for the cases to be considered in this paper.

THEOREM: Let $e_i=0$ represent the equation

$$e_i=a_{i1}x_1+a_{i2}x_2+a_{i3}x_3+a_{i4}x_4+\cdots+a_{in}x_n+c_i=0,$$

where a_{ij} is the coefficient of the unknown x_j and c_i is a constant. Then the systems of equations

$$\begin{array}{ll}
e_1=0 & e_1=0 \\
e_2=0 & e_2=0 \\
e_3=0 \text{ and } & e_3=0 \\
\cdots & \cdots \\
e_{i-1}=0 & e_{i-1}=0 \\
e_i=0 & k_1e_1+k_2e_2+\cdots+k_ie_i+\cdots+k_me_m=0 \\
e_{i+1}=0 & e_{i+1}=0 \\
\cdots & \cdots \\
e_m=0 & e_m=0
\end{array}$$

are equivalent for any constants k_1, $k_2, \cdots k_m$ provided $k_i \neq 0$.

Remark: This theorem says that one may replace any one of a given set of equations by a linear combination of equations of the set so long as the equation replaced is present in a nontrivial manner in the combination. The theorem justifies our obtaining new equations in the manner we have in the examples and asserting that the solutions of the new set are identical with those of the given equations.

The proof of the preceding theorem is not difficult. It is obvious that any set of values for $x_1, x_2, x_3, \cdots x_n$ satisfying the given set of equations also satisfies the set with the ith equation replaced by the combination. Now assume each of the equations in the second set is satisfied. Since each equation, except perhaps $e_i=0$, is satisfied, *i.e.*, $e_1=e_2=\cdots=e_{i-1}=e_{i+1}=\cdots=e_m=0$, the ith equation $k_1e_1+k_2e_2+\cdots+k_ie_i+\cdots+k_me_m=0$ reduces to $k_ie_i=0$ where $k_i \neq 0$ by hypothesis. But then one of our fundamental postulates for the real number system asserts that $e_i=0$, that is, $a_{i1}x_1+a_{i2}x_2+\cdots+a_{in}x_m=0$.

We are now in a position to introduce the refinements promised earlier. Note that we could represent the equations

$$\begin{aligned}
x+2y-3z&=6 \\
2x+5y-2z&=4 \\
x+3y-\ z&=0
\end{aligned}$$

by the rectangular array

$$\begin{array}{cccc}
1 & 2 & -3 & 6 \\
2 & 5 & -2 & 4 \\
1 & 3 & -1 & 0.
\end{array}$$

All we have done is to omit the variables and the equal signs. The set

$$\begin{aligned}
x+2y-3z&=6 \\
y+4z&=-8 \\
-2z&=2
\end{aligned}$$

can be represented by

$$\begin{array}{cccc}
1 & 2 & -3 & 6 \\
0 & 1 & 4 & -8 \\
0 & 0 & -2 & 2
\end{array}$$

where we introduce the zeros to indicate zero coefficients. Notice that the last row in this array considered as an equation is equivalent to the equation $-2z=2$. We now perform several additional operations on this array. First we divide the third row by -2 and obtain

$$\begin{array}{cccc}
1 & 2 & -3 & 6 \\
0 & 1 & 4 & -8 \\
0 & 0 & 1 & -1.
\end{array}$$

Let us introduce the notation R_2-4R_3 and have it mean row two plus -4 times row three. Then the above equations may be transformed by this and the other indicated operations in the following way:

$$\begin{bmatrix}1&2&-3&6\\0&1&4&-8\\0&0&1&-1\end{bmatrix}\rightarrow\begin{bmatrix}1&2&0&3\\0&1&0&-4\\0&0&1&-1\end{bmatrix}\begin{matrix}R_1+3R_3\\R_2-4R_3.\end{matrix}$$

The brackets inserted have no special significance. Let us perform one more operation on this array:

$$\begin{bmatrix}1&2&0&3\\0&1&0&-4\\0&0&1&-1\end{bmatrix}\rightarrow\begin{bmatrix}1&0&0&11\\0&1&0&-4\\0&0&1&-1\end{bmatrix}R_1-2R_2.$$

From the final array of numbers one can immediately read the answers $x=11$, $y=-4$, $z=-1$.

We shall now use the same method on four equations. In order to show what the numbers represent at each stage, we shall also write the corresponding equations. The equations are merely for the sake of clarifying the procedure and are not necessary. In further examples they will be omitted.

It will be convenient to introduce the mathematical name applied to rectangular arrays such as we have used. They are called "matrices," the singular being "matrix." The transformations or operations performed on these matrices will be indicated by the shorthand notation used above. We shall be more detailed than is necessary in practice.

EQUATIONS	CORRESPONDING MATRIX
$3x+2y-\ z+\ w=0$	$\begin{bmatrix}1&-2&2&-1&16\\3&2&-1&1&0\\-2&-1&1&2&-2\\4&-2&-2&3&5\end{bmatrix}$
$-2x-\ y+\ z+2w=-2$	
$x-2y+2z-\ w=16$	
$4x-2\ -2z+3w=5.$	

Note that the third equation has been moved to the first position in the matrix. The reason for this will become apparent in the following steps. As we shall see later, it is not absolutely necessary to begin with a 1 in the upper left-hand corner.

$$\begin{matrix}x-2y+\ 2z-\ w=16\\8y-\ 7z+4w=-48\\-5y+\ 5z\ \ \ =30\\6y-10z+7w=-59\end{matrix}\leftrightarrow\begin{bmatrix}1&-2&2&-1&16\\0&8&-7&4&-48\\0&-5&5&0&30\\0&6&-10&7&-59\end{bmatrix}\begin{matrix}\\R_2-3R_1\\R_3+2R_1\\R_4-4R_1\end{matrix}$$

Next we divide the third row by -5 and place the result in the second row, giving us

$$\begin{matrix}x-2y+\ 2z-\ w=16\\y-\ z\ \ \ =-6\\8y-\ 7z+4w=-48\\6y-10z+7w=-59\end{matrix}\leftrightarrow\begin{bmatrix}1&-2&2&-1&16\\0&1&-1&0&-6\\0&8&-7&4&-48\\0&6&-10&7&-59\end{bmatrix}$$

$$\begin{matrix}x\ \ \ \ \ \ -\ w=4\\y-\ z\ \ \ =-6\\z+4w=0\\-4z+7w=-23\end{matrix}\leftrightarrow\begin{bmatrix}1&0&0&-1&4\\0&1&-1&0&-6\\0&0&1&4&0\\0&0&-4&7&-23\end{bmatrix}\begin{matrix}R_1+2R_2\\\\R_3-8R_2\\R_4-6R_2\end{matrix}$$

$$\begin{matrix}x\ \ \ \ \ \ -\ w=4\\y\ \ \ +4w=-6\\z+4w=0\\w=-1\end{matrix}\leftrightarrow\begin{bmatrix}1&0&0&-1&4\\0&1&0&4&-6\\0&0&1&4&0\\0&0&0&1&-1\end{bmatrix}\begin{matrix}\\R_2+R_3\\\\\dfrac{(R_4+4R_3)}{23}\end{matrix}$$

$$\begin{matrix}x\ \ \ \ \ \ \ \ =3\\y\ \ \ \ \ \ =-2\\z\ \ \ \ =4\\w=-1\end{matrix}\begin{bmatrix}1&0&0&0&3\\0&1&0&0&-2\\0&0&1&0&4\\0&0&0&1&-1\end{bmatrix}\begin{matrix}R_1+R_4\\R_2-4R_4\\R_3-4R_4\\\\\end{matrix}$$

These values for x, y, z, and w are the solutions of the original equations.

Now let us eliminate unknowns by the same procedure but in a less systematic manner. The first matrix represents the equations in the order first given.

$$\begin{bmatrix} 3 & 2 & -1 & 1 & 0 \\ -2 & -1 & 1 & 2 & -2 \\ 1 & -2 & 2 & -1 & 16 \\ 4 & -2 & -2 & 3 & 5 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 2 & -1 & 1 & 0 \\ -8 & -5 & 3 & 0 & -2 \\ 4 & 0 & 1 & 0 & 16 \\ -5 & -8 & 1 & 0 & 5 \end{bmatrix} \begin{matrix} \\ R_2-2R_1 \\ R_3+R_1 \\ R_4-3R_1 \end{matrix}$$

$$\begin{bmatrix} 3 & 2 & -1 & 1 & 0 \\ -20 & -5 & 0 & 0 & -50 \\ 4 & 0 & 1 & 0 & 16 \\ -9 & -8 & 0 & 0 & -11 \end{bmatrix} \begin{matrix} \\ R_2-3R_3 \\ \rightarrow \\ R_4-R_3 \end{matrix} \begin{bmatrix} 3 & 2 & -1 & 1 & 0 \\ -42 & 1 & 0 & 0 & -128 \\ 4 & 0 & 1 & 0 & 16 \\ -9 & -8 & 0 & 0 & -11 \end{bmatrix} 3R_3-2R_4$$

$$\begin{bmatrix} 3 & 2 & -1 & 1 & 0 \\ -42 & 1 & 0 & 0 & -128 \\ 4 & 0 & 1 & 0 & 16 \\ -345 & 0 & 0 & 0 & -1035 \end{bmatrix} \leftrightarrow \begin{matrix} 3x+2y-z-w=0 \\ -42x+\ y\ \ \ \ \ \ =-128 \\ 4x+\ z\ \ \ \ \ \ =16 \\ -345x\ \ \ \ \ \ \ \ =-1035. \end{matrix} \quad R_4+8R_2$$

Solving these equations for x, y, z, w, beginning with the last, we get the same answers as before. What we have above is one equation in one unknown, one in two unknowns, one in three unknowns, and one in four unknowns. These can be solved by substitution.

Suppose we were faced by a set of equations such as

$$\begin{aligned} .345x+.276y-.021z+.123w&=1.270 \\ -.125x-.342y+.123z-.562w&=\ .004 \\ .504x-.252y-.311z+.245w&=\ .672 \\ .024x+.425y+.522z-.322w&=\ .422. \end{aligned}$$

Then the second procedure used above would be difficult to apply, even if one had a desk calculator at one's disposal. However, with a desk calculator available, the first procedure of obtaining one's along the diagonal can be systematically accomplished. One merely divides each term in the first equation by .345. This will result in a coefficient of one for x. By using the obvious multiples of this new first equation in combination with the remaining equations, one eliminates x from them. We then have a system which can be represented by

$$\begin{bmatrix} 1 & .080 & -.006 & .036 & 3.680 \\ 0 & -.332 & .124 & -.558 & .464 \\ 0 & -.292 & -.281 & .227 & -1.178 \\ 0 & .423 & .522 & -.323 & .334 \end{bmatrix} \begin{matrix} \\ R_2+.125R_1 \\ R_3-.504R_1 \\ R_4-.024R_1. \end{matrix}$$

We continue by dividing the second equation by $-.332$, obtaining a 1 on the diagonal. Using this 1, it is not difficult—merely tedious —to get 0's above and below it, i.e., eliminating y from all equations but the second. Continuing in this manner, we generate 1's on the diagonal with the numbers in the last column being the solutions sought. Such solutions will, in general, be only approximate and not exact solutions. We shall not go into the problem of rounding approximate numbers here.

The preceding method is especially advantageous in the case of dependent or inconsistent equations, for in each of these cases the determinant of the coefficients is equal to zero. Professor MacDuffee, in the article referred to previously, treated a system of dependent equations. But we shall also give an example of each of these types in order to make our discussion more complete.

We now solve the system:

$$\begin{aligned} x+3y-z-\ w&=4 \\ 2x+4y-z+\ w&=-1 \\ 3x+\ y+z+\ w&=0 \\ x-\ y+z-2w&=7 \end{aligned} \leftrightarrow \begin{bmatrix} 1 & 3 & -1 & -1 & 4 \\ 2 & 4 & -1 & 1 & -1 \\ 3 & 1 & 1 & 1 & 0 \\ 1 & -1 & 1 & -2 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 & -1 & -1 & 4 \\ 0 & -2 & 1 & 3 & -9 \\ 0 & -8 & 4 & 4 & -12 \\ 0 & -4 & 2 & -1 & 3 \end{bmatrix} \begin{matrix} \\ R_2-2R_1 \\ R_3-3R_1 \\ R_4-R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & -1 & -1 & 4 \\ 0 & -2 & 1 & 3 & -9 \\ 0 & 0 & 0 & -8 & 24 \\ 0 & 0 & 0 & -7 & 21 \end{bmatrix} \begin{matrix} \\ \\ R_3-4R_2 \\ R_4-2R_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 3 & -1 & -1 & 4 \\ 0 & 1 & -1/2 & -3/2 & 9/2 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix} \begin{matrix} \\ \frac{R_2}{(-2)} \\ \frac{R_3}{(-8)} \\ \frac{R_4}{(-7)} \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1/2 & 7/2 & -19/2 \\ 0 & 1 & -1/2 & -3/2 & 9/2 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1-3R_2 \\ \\ \\ R_4-R_3 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1/2 & 0 & 1 \\ 0 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1-(7/2)R_3 \\ R_2+(3/2)R_3 \\ \\ \end{matrix}$$

The last matrix shows this is a special kind of dependent system of equations. For, converting the rows to equations, we have $w=-3$, $y-\frac{1}{2}z=0$, and $x+\frac{1}{2}z=1$. Suppose we let x equal the parameter t. Then our solutions become $x=t$, $z=2-2t$, $y=\frac{1}{2}z=1-t$, $w=-3$. In this instance w does not depend on any parameter and must be given the value -3 only. This is a dependent set of equations with infinitely many sets of numerical values possible for x, y, and z.

Whenever the number of unknowns and the number of equations are the same and the equations are dependent, at least one row of zeros will appear in the matrix if the process is continued long enough. For all practical purposes we could have obtained the solutions from the third matrix in the sequence.

We present the following example of an inconsistent set:

$$\begin{matrix} x+\ y+2z-\ \ w=\ 2 \\ 3x-2y+\ z-\ 2w=\ 8 \\ 6x-7y+7z-13w=14 \\ 2x+\ y-\ z+\ 3w=\ 0 \end{matrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & -1 & 2 \\ 3 & -2 & 1 & -2 & 8 \\ 6 & -7 & 7 & -13 & 14 \\ 2 & 1 & -1 & 3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & -1 & 2 \\ 0 & -5 & -5 & 1 & 2 \\ 0 & -13 & -5 & -7 & 2 \\ 0 & -1 & -5 & 5 & -4 \end{bmatrix} \begin{matrix} \\ R_2-3R_1 \\ R_3-6R_1 \\ R_4-2R_1 \end{matrix}$$

$$\downarrow$$

Moving last row to second row and dividing it by -1:

$$\begin{bmatrix} 1 & 1 & 2 & -1 & 2 \\ 0 & 1 & 5 & -5 & 4 \\ 0 & -5 & -5 & 1 & 2 \\ 0 & -13 & -5 & -7 & 2 \end{bmatrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 5 & -5 & 4 \\ 0 & 0 & 10 & -12 & 11 \\ 0 & 0 & 60 & -72 & 54 \end{bmatrix} \begin{matrix} R_1-R_2 \\ \\ (R_3+5R_2)/2 \\ R_4+13R_2 \end{matrix}$$

$$\downarrow$$

$$\begin{bmatrix} 1 & 0 & -3 & 4 & -2 \\ 0 & 1 & 5 & -5 & 4 \\ 0 & 0 & 10 & -12 & 11 \\ 0 & 0 & 0 & 0 & -12 \end{bmatrix} \begin{matrix} \\ \\ \\ R_4-6R_3. \end{matrix}$$

We carry the process no further since, the last row indicates the equations are inconsistent; for it reads

$$0x+0y+0z+0w=-12,$$

which is a contradiction.

The foregoing method can also be used with efficiency to determine the nature of the solutions of sets of equations such as

$$\begin{matrix} 2x-2y+4z=6 \\ 3x-2y+2z=4 \\ x-\ y+3z=0 \\ -2x+\ y-\ z=2. \end{matrix}$$

The corresponding matrix is:

$$\begin{bmatrix} 2 & -2 & 4 & 6 \\ 3 & -2 & 2 & 4 \\ 1 & -1 & 3 & 0 \\ -2 & 1 & -1 & 2 \end{bmatrix}.$$

To make use of the 1 in the third row, first column, we move the third row to the first row position. The other rows may be recopied in any order.

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 2 & -2 & 4 & 6 \\ 3 & -2 & 2 & 4 \\ -2 & 1 & -1 & 2 \end{bmatrix}$$

The reason for obtaining the 1 in the upper left hand corner is the resulting ease in converting the numbers below it to 0's, i.e., eliminating x's from the other equations. If no equation contains a coefficient of 1 for x, we would divide one of the equations by the coefficient of x in that equation and so manufacture a 1. Fractions resulting from such a division should not be approximated by decimals if the coefficients are exact numbers. We now obtain in our present example the matrix

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 0 & -2 & 6 \\ 0 & 1 & -7 & 4 \\ 0 & -1 & 5 & 2 \end{bmatrix} \begin{matrix} \\ R_2-2R_1 \\ R_3-3R_1 \\ R_4+2R_1. \end{matrix}$$

Comparing the multiples of R_1 used with the entries in the first column in the preceding matrix should make clear the origin of these multipliers. We now interchange the second and third rows in order to get a one in the second row, second column position.

$$\begin{bmatrix} 1 & -1 & 3 & 0 \\ 0 & 1 & -7 & 4 \\ 0 & 0 & -2 & 6 \\ 0 & -1 & 5 & 2 \end{bmatrix}$$

We are now ready to eliminate y's from all equations but the second.

$$\begin{bmatrix} 1 & 0 & -4 & 4 \\ 0 & 1 & -7 & 4 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -2 & 6 \end{bmatrix} \begin{matrix} R_1+R_2 \\ R_2 \\ \hline -2 \\ R_4+R_2 \end{matrix}$$

We are now ready to eliminate z's from all equations but the third.

$$\begin{bmatrix} 1 & 0 & 0 & -8 \\ 0 & 1 & 0 & -17 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} R_1+4R_3 \\ R_2+7R_3 \\ \\ R_4+2R_3 \end{matrix}$$

This last matrix indicates the four equations are indeed consistent and have as solutions $x=-8$, $y=-17$, $z=-3$.

The above method also works efficiently on systems having the number of unknowns exceeding the number of equations. In such a case the equations are always dependent. For example,

$$\begin{matrix} 3x-2y+\ z=4 \\ 2x-\ y+3z=-1 \end{matrix} \leftrightarrow$$

$$\begin{bmatrix} 3 & -2 & 1 & 4 \\ 2 & -1 & 3 & -1 \end{bmatrix} \leftrightarrow$$

$$\begin{bmatrix} 1 & -1 & -2 & 5 \\ 2 & -1 & 3 & -1 \end{bmatrix} \begin{matrix} R_1-R_2 \\ \leftrightarrow \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & -2 & 5 \\ 0 & 1 & 7 & -11 \end{bmatrix} R_2-2R_1.$$

We have eliminated x from the second equation. We notice that we could now solve the second equation for either y or z. Before doing this let us perform one more operation.

$$\begin{bmatrix} 1 & 0 & 5 & -6 \\ 0 & 1 & 7 & -11 \end{bmatrix} R_1+R_2$$

These equations now say $x=-6-5z$ and $y=-11-7z$, so that we could let z act as a parameter. If we wanted to express the entire set of solutions in parametric form, they would be

$$x=-6-5t,\ y=-11-7t,\ z=t.$$

Just as there are theorems in the theory of determinants specifying relations between values of certain determinants and solutions of a system of equations, so there are theorems relating the solutions of a set of equations to the nature of the final matrices encountered above. Actually one relatively brief theorem sums up the situation; however, the theorem can be stated in compact form only after other concepts involving matrices have been introduced. Fortunately, knowledge of the theory is not necessary in making practical use of the procedure. Whenever solutions exist, their values are also immediately evident. Likewise, when solutions do not exist, this is also obvious.

The reader desiring additional information concerning this topic is referred to some recent college level texts:

Andree, Richard V., *Selections from Modern Abstract Algebra*. New York: Henry Holt and Company, 1958.

Hohn, Franz E., *Elementary Matrix Algebra*. New York: The Macmillan Company, 1958.

Mac Duffee, C. C., *Theory of Equations*. New York: John Wiley & Sons Inc., 1954.

No space geometry in the space age?

CHARLES H. SMILEY, *Brown University, Providence, Rhode Island, and* DAVID K. PETERSEN, *North High School, Omaha, Nebraska.*

In what way should the high school mathematics curriculum be revised because of man's increased interest in outer space?

ON OCTOBER 4, 1957, the Soviet Union launched the first artificial satellite and thereby set the stage for a thorough review of American education. Every potential critic of education came forth and spoke his piece. In the interval which followed, there was nothing right about American educational methods. The courses taught in secondary schools and in colleges were all the wrong ones, trivial at best and positively harmful at worst. Some critics even suggested that we must get busy and imitate the Russians who had, at least in part, been imitating us. "More science and less liberal arts" was a common phrase. A "crash program in science education" was recommended frequently.

Luckily, sanity returned, and our scholars were able to set about a thorough, carefully considered examination of our educational machinery. Some groups had already started work in this area. The most noteworthy example, of course, was the formation of the Commission on Mathematics headed by Professor Albert W. Tucker of Princeton University. This Commission was created in 1955 to review the existing secondary school curriculum. The foresight of the College Entrance Board in the pre-Sputnik era is certainly to be commended; now that the report of the Commission is available, one can find little fault. One is pleased to find more emphasis placed on mathematical structure, less on rote memory and the proofs of large numbers of theorems in plane geometry. There is one recommendation which should be reconsidered, however. With man taking his first cautious steps out into space, it was decided that a separate course in space geometry was no longer to be recommended. Some spatial concepts are to be included in the plane geometry, and that is all. Will this give our young scientists the background, the training, and the understanding they need?

Is not this move to drop space geometry a step backward rather than a step forward? Should we not instead revise, improve the course in space geometry, increase the emphasis on the visualization of three-dimensional configurations? Should we not consider the possibility of formulating an improved course in space geometry with a good deal of work in astronomy?

Having raised the questions, may we offer some specific recommendations? For one thing, one can use the motions of an artificial earth satellite to help a student develop the ability to visualize a movement in three dimensions. As strange as it might seem to anyone with even a slight intuitive notion of three-dimensional space, one hears often the question, "If a satellite goes by in the morning, traveling from southwest to northeast, must it not go back in the evening on a northeast-to-southwest course?" Of course it is simple enough to draw two circles with a line across each (Fig. 1) and to point out that if the circle at the left is turned over, one might see something like the circle at the right. It is not hard then to offer the two circles as two-dimensional pictures of the two halves of a single sphere, and the orbit of a satellite moving about it.

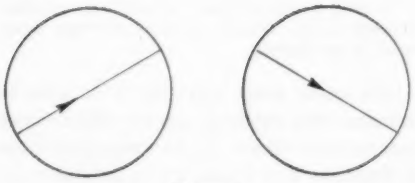


Figure 1

One might then speak of the rotation of the earth under the orbit of the satellite and explain why it is so much easier to launch an east-bound satellite than a west-bound one. (The earth's rotation helps to provide some of the kinetic energy needed for an east-bound one!) One could ask whether the Soviet scientists had a more difficult task or an easier one than our own men. How does latitude figure in this? A little later, one might even consider satellites high enough above the surface of the earth so that, though they are moving eastward, they appear to a person on the earth's surface to be moving westward because the period of revolution of the satellite is longer than the period of rotation of the earth. Will the observer on the earth be moving at a higher speed than the satellite, and if not, how can he keep passing it?

The brightest students might be helped to understand what happens when the satellite begins to lose energy due to atmospheric friction, and consequently speeds up in its motion around the earth. The orbit becomes smaller but the speed in the path increases also, a fact which will not be obvious to the uninformed. "But it must slow down!" is the common argument.

In the study of eclipses lies a wealth of material useful not only for three-dimensional visualization but also in comprehending space motion. The very brief and incomplete description of the eclipse situation which follows should make this evident.

The shadow cast by the earth from the sun's rays is called the *umbra* (Fig. 2). This cone-shaped shadow is the area with no light received from the sun. The area of

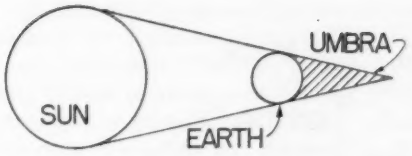


Figure 2

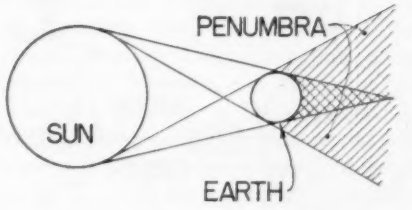


Figure 3

diminished sunlight is the *penumbra* (Fig. 3). When the moon passes through the umbra, obviously it is cut off from the sun and we have a lunar eclipse. Likewise, when the moon passes between the sun and the earth, the shadow cone of the moon produces a solar eclipse to an observer on earth.

This would seem very elementary until we consider that there is more affecting the eclipse situation than shadow cones. We must also consider the plane of the ecliptic (apparent path of sun among the stars), the plane in which the moon moves about the earth, and relative distances of bodies involved at the time eclipse conditions are satisfied. For example, it is only when the line of nodes (line of intersection of plane of moon's orbit and the plane of the ecliptic) very nearly coincides with the earth's radius vector that an eclipse can occur (Fig. 4). Otherwise, the umbra will pass above or below the earth or moon. In the case of solar eclipses, the situation is further complicated by the fact that the distance of the moon from the earth is such that the shadow cone of the moon does not always reach the earth, resulting in the observer on earth being placed in the shadow beyond the apex of the shadow cone (annular eclipse). (See Figure 5.)

Notice what has been included in this brief discussion—cones, planes, space vec-

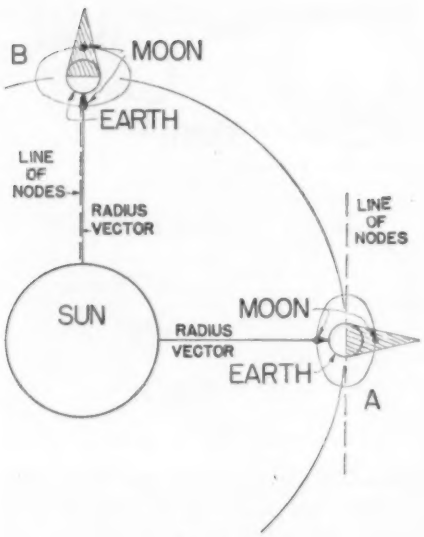


Figure 4

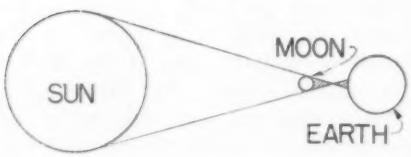
When the moon is on the line of nodes, it is in the same plane as the sun and earth (plane of the paper). In position *A*, no eclipse would be produced, since the moon would not be in line with the sun and earth when it passes the line of nodes. However, in position *B*, when the moon passes the line of nodes (is in the same plane as the sun and earth), it is also either between the sun and earth or directly opposite the earth from the sun (on the earth's radius vector). Hence, an eclipse would occur at position *B*.

tors, space distance, and space movement. Is this not a meaningful way of looking at three-dimensional space? How well can you visualize these concepts? Try these:

Let us assume you could watch an eclipse in Providence, R. I., Nome, Alaska, or on the equator. Other conditions being equal, which would have the longest duration of totality? Why? (Consider the path of the moon's shadow on the earth.)

There is an annular eclipse. What would you see?

Figure 5



During a solar eclipse, would it be possible to observe the moon's shadow moving westward? If so, explain.

One needs some training to be able to visualize two paths of planets which come near to each other—as, for example, those of Neptune and Pluto. Pluto at one point in its path may actually be nearer the sun than Neptune. This can be seen clearly if one has a stereoscopic picture of the two paths. But one can lay a good groundwork for the problem by considering the simpler one of finding the shortest distance between two skew lines in space. One can start by asking, "Is there a line in space which will be perpendicular to each of the two given skew lines?" If the student can't picture the situation in his mind, you can ask him, "If the lines were wires, could you put a loop around the two wires, and then if you did, could you tighten the loop before tying? How would the rope slide? Where would it stop?" If the student doesn't like this approach, let him try his hand!

Another device to assist in our attempt at better space comprehension is to examine the seasons. Here is an area that certainly commands universal interest accompanied by many misconceptions of the "why" involved.

The distance of the earth from the sun is not the primary reason for the seasons as is often thought; in the northern summer, the earth is farther from the sun. Since the seasons are determined by the position of the earth's axis with reference to the sun, we can take advantage of the space concepts involved in this situation—the intersection of the plane passing through the earth's equator (celestial equator) and the plane of the ecliptic, the effect of the rotation of the tilted earth at various positions of the orbit on the amount of light received, and the effect of light rays striking a spherical surface (Fig. 6). Undoubtedly, many students would like to pursue this further into related areas. How do the seasons in the southern hemisphere compare with ours? What is the meaning of the

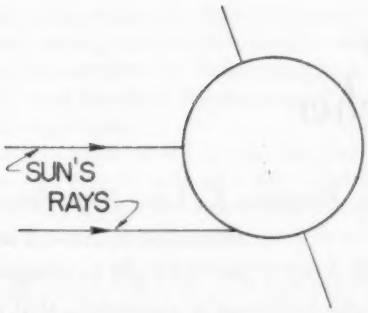


Figure 6

climatic zones learned about in geography courses? Some students might like to tie up movements of the earth about the sun with our calendar and the history of how it evolved.

In the foregoing discussion, we are not looking for complete agreement on the part of the reader. The areas discussed are only typical of untapped resources which should be of great benefit to high school students. They are not necessarily the best ones. Our thinking on what should be included in such a modernized course is as yet tentative. Our concern over a decreased offering of space-comprehension material is, however, not the least bit tentative. This era in history demands that any modernizing in the curriculum encompass a shift toward increased space study. We think the answer is the formulation of a mathematically-designed astronomy course with heavy emphasis on space geometry.

What's new?

BOOKS

College

Information and Error, Solomon Diamond. New York: Basic Books, Inc., 1959. Cloth, xii+307 pp., $5.

Intermediate Algebra for the College Student, J. Vincent Robison. Princeton, New Jersey: D. Van Nostrand Company, Inc., 1959. Cloth, vii+286 pp., $4.25.

Miscellaneous

Echoes of Bats and Men, Donald R. Griffin. New York: Doubleday and Company, Inc., 1959. Paper, 156 pp., 95¢.

How Old is the Earth, Patrick M. Hurley. New York: Doubleday and Company, Inc., 1959. Paper, 160 pp., 95¢.

Magnets, Francis Bitter. New York: Doubleday and Company, Inc., 1959. Paper, 155 pp., 95¢.

The Neutron Story, Donald J. Hughes. New York: Doubleday and Company, Inc., 1959. Paper, 158 pp., 95¢.

Purchase Guide for Programs in Science, Mathematics and Modern Foreign Languages, Council of Chief State School Officers. Boston: Ginn and Company, 1959. Paper, vii+336 pp., $3.95.

Soap Bubbles and the Forces Which Mould Them, C. V. Boys. New York: Doubleday and Company, Inc., 1959. Paper, 156 pp., 95¢.

BOOKLETS

Secondary

Math Fun, Ray Skotte and Yngve Magnuson. Minneapolis: Published by the authors, 6380 Monroe Street, Minneapolis 21, 1959. Paper, 88 pp.

Modern Mathematics Topics and Problems, Daymond J. Aiken and Charles A. Beseman. New York: McGraw-Hill Book Company, Inc., 1959. Paper, iv+140 pp., $1.48.

Nomograms, C. V. Gregg. Doncaster, England: Mathematical Pie Ltd., 1959. Paper, 23 pp., 20¢.

MODELS

String models of the quartic, intersecting cylinders, hyperbolic paraboloid, and right helicoid. Manufactured by the Educational Division of Industrial Research Laboratories, Inc., P.O. Box 471, Hempstead, New York.

Some heretical thoughts from an orthodox teacher

WALLACE MANHEIMER, *Franklin K. Lane High School, Brooklyn 8, New York.*

"The classroom teacher has the right to disagree, of course, with the currently fashionable prediction that the mathematics curriculum in the secondary schools will be altogether different in twenty years."

FELLOW TEACHERS, do you feel pushed around? Can you turn your head without thinking that you are starting to generate a group of rotations? Can you call your daughter a rag, a bone, and a hank of hair without amending it to, "a set, $\{x\}$, determined by the relation, x is a rag, or a bone, or a hank of hair"? Can you settle down to your Friday night poker game without reflecting that you are about to plot a point set in a probability space? Are some of your colleagues more supercilious about disjunction, conjunction, and truth values in September than they were in May?

More seriously, do you feel that your pupils are cheated by having such a reactionary teacher as yourself? That they will not become members of the cognoscenti; that they are learning mathematics that is 600 years old? Perhaps you do. Surely the newspapers and magazines have been telling you so for a long time.

You may have wondered how your pupils are going to absorb this new material, and how they can relate it to traditional subject matter. In any case, you must expect that they will face it, for "modern mathematics" is all the rage. The movement to teach it in the secondary schools has become the latest panacea for the ills of our subject. Unlike other movements of the past it is fortified by considerable college and university backing, widespread publicity, and large sums of money.

A few words of caution have been thrown in the path of the modern mathematics steamroller by educators and mathematicians. A notable one by Dr. Morris Kline[1] appeared in the pages of THE MATHEMATICS TEACHER. However, little has been heard from classroom teachers, who are asked to put the program into practice. Perhaps we are abashed in the face of so much erudition. Nevertheless, as one of the teaching fraternity, I would like to express a viewpoint that might be similar to that of many of my silent colleagues. I shall dispense with a summary of the outlines of the modern mathematics movement, for it is sufficiently clear to all of us by now.

WILL THE MOVEMENT MAKE MATHEMATICS "DESCRIPTIVE"?

Mathematics teachers take pride in the fact that their subject is analytical rather than merely descriptive. For example, in the study of quadratic equations we are not content merely to *define* them, utter some remarks *about* them, and show how some natural phenomena afford *illustrations* of them. We prefer to penetrate more

[1] Morris Kline, "The Ancients versus the Moderns, A New Battle of the Books," THE MATHEMATICS TEACHER, LI (October, 1958), pp. 418–427.

deeply: to analyze their graphs, study the discriminant, probe relationships among roots and coefficients. To borrow a fashionable word, we study the *structure* of quadratic equations.

It should be noted in passing that an effect of this is to make the quadratic equation a weapon for the solution of other mathematical and scientific problems.

Consider, now, the proposal for a high school study of some of the theories of sets, groups, algebraic fields, rings, and ideals. Let us agree that these concepts are vital to the structure of algebra. But can pupils who are only about ready to absorb the structure of quadratic equations possibly obtain more than a superficial and descriptive understanding of them? Isn't it putting the cart before the horse to study the connective tissue of algebra before one knows what it is that is being connected?

A question or two may clarify the objection. Are pupils able to use these concepts as weapons in the solution of other mathematical and scientific problems? Are they really learning more about sets, groups, and so on, than a mere description of them? It might be helpful to take a typical textbook in group theory and find out what part of it other than the definition of a group we are planning to teach in the high schools. If the answers to these questions are not encouraging we must face the fact that our pupils will have merely a descriptive, superficial, and sterile knowledge of "modern mathematics."

Consider the history of group theory and set theory. The discovery of group theory was as a weapon for the study of roots of equations of degree higher than four. They were later used as a tool to connect apparently dissimilar geometric spaces. Set theory was developed as a weapon to underpin and later to generalize the infinite processes of the calculus. Of course they are fascinating disciplines of their own, but a chief function was to serve as mathematical tools, and study of them was thorough and analytical.

This is not to claim that we must be slaves to history. Many of man's century-long intellectual struggles are today telescoped onto brief studies. But there is a limit, and it is overreached when functional and purposeful aspects have been removed and mere description of the fruits remains.

What an irony it would be if set theory and group theory, which have integrated vast stretches of mathematics, were to give our immature pupils a disjointed and fragmentary view of the subject!

Does the movement violate educational principles?

Let us analyze the movement to modern mathematics in the light of widely accepted educational principles. One principle is that learning proceeds most naturally from the concrete and particular to the abstract and general. Although to a lesser degree, this applies to brilliant people as well as dullards. It is unfortunately easily forgotten by mathematics teachers who—sitting on our little pinnacles of learning and skill—sometimes forget about the underbrush we had to clear away to get there. In some cases we forget, sadly enough, because it has been a while since *we* learned any new mathematics. Try yourself out on a difficult problem, for example, in the theory of numbers. Won't you often "get the hang" of the problem by trying out specific numerical cases?

Our students are more bound by this restriction of approaching the abstract by way of the concrete, because they are less mature than we are. But the theories of sets, groups, and other mathematical structures are the very apotheoses of abstraction. They are the view from the mountain tops, reached after centuries of struggle. Can we seriously expect our youngsters, who are still grappling in the underbrush, to get any of the richness and depth of these concepts?

Another principle is that learning is more meaningful when it proceeds from

the known to the unknown. It is too obvious for elaboration, but a relevant illustration is in order.

Traditional Euclidean geometry is coming under a good deal of fire these days because it has so many logical imperfections. It is contrasted unfavorably with a study of space via co-ordinates. But before we scrap this two-thousand-year-old approach to physical space, let us weigh its advantages. Perhaps the creaky old Euclidean development has shown such hardiness not because of excellence as mathematics, but because it provides a remarkably simple transition from sense data to *some* sort of postulational system. It is a magnificent jump from the known to the unknown. Watch the face of a pupil who discovers that the equality of vertical angles—a concept he has accepted ever since he first used a pair of scissors—depends upon the subtraction axiom. He has learned not only an important mathematical lesson, but an important lesson in the value of mathematics to mankind. He is not ready to comprehend our dissatisfaction with the system as a whole. And if we were to shore it up with a number of rather subtle and abstract postulates on order, we might save the system but lose the pupil!

Another principle is one that teachers do not boast about but use every day. It is that we should not take a hard and fast view about rationalizing every concept that we teach. In fact, when the reason *why* something is true is very much more difficult than its applications, we often jettison, or at least postpone, the explanation. I think we can justify this practice on the grounds of a sort of intuition we have built up about what the pupil can or cannot absorb naturally enough so as not to impede the progress of the main work.

An illustration. Last year I attempted to teach the concept of equivalent equations: equations having the same root or roots, to my ninth-year algebra class. The children listened willingly, even politely. They dutifully learned the rubric, "If the same number is added to each side of an equation the result will be an equivalent equation." However, a week after we had moved on to other things, the concept of equivalent equations had dimmed. It had become, "You can add the same thing to both sides." Nevertheless, they were solving equations happily, and eager to move on and apply them. I concluded that the pupils were not ready for this abstraction, and that it was not necessary for their immediate progress in the work.

This may have been the result of poor teaching. I certainly mean to try it again. However, I shall not be surprised if the class yawns at my efforts. Haven't we all had the experience of finding that a concept that appears overwhelmingly cogent to us sits very poorly on our children? The human mind and the learning process are complicated things, and they are not too susceptible to *a priori* assumptions.

By the way, if a concept as simple as equivalent equations can discourage an experienced teacher, what will an attempt to teach rings and ideals do to an inexperienced one?

Oil on troubled waters

A word or two to my colleagues, those charging under the new banner as well as those teaching "600-year-old mathematics." Let's not get into a lather about this issue. Classroom teachers like myself must realize that the ferment of new ideas is good for the teaching of mathematics. When all the smoke has cleared away, there will doubtless be improvements, many of which we do not foresee at present.

To the innovators: please spare our feelings. We 600-year-oldsters are doing our bit too. After all, nobody seriously doubts the tremendous current usefulness of traditional algebra and geometry. They may not corral pi mesons, but they still weigh potatoes and build schoolhouses! As to the appreciation of structure and other esthetic values, surely these do not tarnish with the age of the subject. Palestrina is 400

years old, but many of us prefer his structures to Schönberg's strictures. Euclid's beloved proof of the infinity of primes must be about to celebrate its 2200th birthday.

A TEACHER'S VIEW

Finally, what is a classroom teacher's view in this debate? First, there is, I think, a different emphasis toward mathematics that grows out of teaching it at our level. We realize that mathematics achieves its great powers through abstraction and generalization. But there must be a sharp and careful discipline over this explosive power of the subject. Mathematics is not a collection of useless theorems drawn from idle postulates. In the hands of its great creators it continually renews a relevance, however subtle, to the panorama of human culture. In a broad sense it is almost a social science.

Also, as Professor Kline indicates, the emphasis on mathematical structure is only one aspect of mathematical creativity. Perhaps history will show that emphasis on structure is a sign of the *consolidation* of mathematical thought to prepare for the leap of another Gauss or Cantor of the future.

From the point of view of the teacher it means that the broadest contact with the real world must be stressed—the real world of our pupils. We must not overrate their maturity and consider them ready for broad vistas when they can reach only to the tips of their fingers.

Of course the teacher must lead his pupils to an ever widening comprehension of mathematical thought. But he must work gradually and in a way that maintains mathematics' greatest cultural values. We certainly want our pupils to use mathematics in their physics, chemistry, and other classes.

Application of this view to elementary algebra, for example, would play down the grand structure of the subject as not sufficiently relevant, at the moment, to our pupils' experience. On the other hand the view would warmly welcome a unit on meaningful applications of inequalities.

Let none of the above be interpreted as abdication of the teacher's responsibility to be a perpetual spur to creativity. He must always throw his pupils on their mettle; he must train them to reach for new ideas. He must carefully nurture their powers of abstraction, but they must leap from ground that is solidly beneath them.

Let pupils become expert in manipulation, for this is a necessary prerequisite to naturalness of thinking in the subject. There is nothing wrong in technical competence. The pupils' college teachers later on may be more grateful for good algebraic technique than for partially digested generalities.

This point of view has its bearing on curriculum research and the training of teachers. To paraphrase, mathematical education is too important to leave to the mathematicians. They need the help of philosophers, psychologists, and just plain teachers.

The classroom teacher has the right to disagree, of course, with the currently fashionable prediction that the mathematics curriculum in the secondary schools will be altogether different in twenty years. May he guess, instead, that actual trial of the new ideas will force something of a retreat upon its proponents? It may be that the residue will simply be the use of some of the *language* of sets in clarifying a number of basic ideas of algebra, such as root and variable. But if this is so the syllabus will be quite recognizable in twenty years, even to the conservative eyes of today.

A SUMMARY

It is dangerous to summarize with an analogy, but I shall take the chance. Suppose two beginners' classes are to be formed for instruction in chess. The first class will be given the simple, petty, fragmentary details of chessboard tactics, such as not placing pieces en prise, queening pawns, simple checkmating combina-

tions, and so on. The second class will be taught the grand strategy of the game, which today is a highly theoretical, structured, and abstract set of principles, such as control of the center, control of time and space, and so on.

I have no doubt that the second class would do far better in a test devised by the second teacher. But what do you think would happen in an inter-class tournament? Don't you agree that the carnage would be frightful? Wouldn't the aggrieved pupils of the "structure" of chess complain that their shiny new concepts were of no use in the stress of battle?

Perhaps the second teacher might even come to the conclusion that these concepts, which are the quintessence of chess to an experienced player, have no more than descriptive value for the player who is a tyro!

Im a not enough of a mathematician—or chess player—to decide upon the aptness of this analogy. I do have a conviction that the pupil who can take a difficult problem on his level and handle it with courage, ingenuity, and resourcefulness is far ahead of one who has merely learned to express simple data in the language of sets and groups.

Have you read?

GERARDI, WILLIAM JOHN, "Modern Mathematics," *Baltimore Bulletin of Education*, November 1958, pp. 25–32.

Much has been and still is being said about "modern mathematics." This is another article on the same topic, but I believe you will profit from reading it.

The author begins with the question, "What is it?" then goes on to who wanted a change, what will the change include, and how will teachers prepare for the change. In answering these questions he gives clear and brief statements of his views on the subject. For example, modern mathematics is logic, statistics, topology, theory of games, information theory, etc. The applied mathematician advocated the change in the curriculum. The new curriculum will emphasize structure, creativity, models, and reasoning. The geometry will be synthetic and analytic, Euclidian and non-Euclidean, as well as plane and solid. Theory of equations, theory of numbers, statistical inference, probability, and elementary analysis will all be a part of secondary school mathematics.

The author tempers all this by saying the transition will be gradual. Read this as another article to help you reach your own decision.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

PRICE, DEREK J. DESOLLA, "An Ancient Computer," *Scientific American*, June 1959, pp. 60–67.

In this age of fabulous electronic computers it is exciting to learn that two thousand years ago a computer was in existence. This article is the story of such a device recovered from two hundred feet of water off the shore of the isle of Antikythera. The computer is still being studied, and an attempt to reconstruct it is being made to find out what it did and how it operated.

You will be interested to read that no reference has been found that indicates the existence of such a machine. It has three dials, it has cog-wheels, and the angles of the teeth are each 60°. Any wheel fits any other wheel. It has a zodiac scale and makes reference to the planets. It is a graduated instrument, which was unknown two thousand years ago. It has an astronomical calendar written by Geminas, who lived about 70 B.C. But what is it and what did it do? You read the article and guess. I am certain you will want to watch for further information about this interesting computer.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

SARNER, DAVID S., and FRYMIER, JACK R., "Certification Requirements in Mathematics and Science," *School Science and Mathematics*, June 1959, pp. 456–460.

This article doesn't show you how to teach mathematics, neither does it give you any interesting mathematical material, but you should all read it. It is a report of a study to determine what minimum requirements must be met to teach mathematics in each of the fifty states. Do you want to guess the range of this minimum? From 0 to 24 semester hours with 18 the mode and 15.1 the median. Are we equipped to teach with this level of preparation? Do some states accept high school mathematics as part of this minimum? Is a particular level of competence required? The authors answer some of these questions and go ahead to discuss the results of the study in terms of whose responsibility it is to establish and carry out adequate preparation programs. He also points out your responsibility. Read the article, think about it, study the situation in your state, and then take the proper action.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

Preparation in mathematics of mathematics teachers

JOHN J. KINSELLA, *New York University, New York, New York.*

What should the mathematics teacher's preparation be for the years after 1963?

THE MATHEMATICAL PREPARATION of secondary school mathematics teachers has become more of a pressing problem within the past five years because of various movements to change the mathematics curriculum of the secondary school as well as the college. Among these movements have been those associated with the University of Illinois Committee on School Mathematics, the Commission on Mathematics of the College Entrance Examination Board, the Maryland junior high school project sponsored by the American Association for the Advancement of Science, and the Committee on the Undergraduate Program of the Mathematical Association of America. *Insights into Modern Mathematics*, a recent yearbook of the National Council of Teachers of Mathematics, gives evidence that the leaders of that organization are also calling for change. If, as a result of experiments with the new proposals, some of the proposed changes become part of the mathematics curriculum, there is little doubt that the preparation of secondary school mathematics teachers will have to be modified.

Like most problems, this one is part of a larger one, namely, what the total preparation of these teachers should be. There is far from complete agreement on the amount of a teacher's education that should be given to the foundations of education and methods courses, to liberal arts, and to specialized subject matter. There is agreement, however, that teachers should be admirable human beings, able conveyors of our culture and civilization, educators in the best professional sense, and specialists in at least one branch of human knowledge. Need I add that this global meeting of the minds is of little help in seeking answers to our special problem?

One way to approach a serious question is to pose several subsidiary ones. Progress in answering these subsidiary questions may bring us closer to the solution of the larger one. I submit these three queries:

1. Which criteria should be used to design the mathematical preparation of secondary school mathematics teachers?
2. How does the present program of mathematical preparation rate with respect to these criteria?
3. How should the present program be modified to implement these criteria better?

I

The question of criteria is the most crucial and difficult one. To deal with it we must first decide what the responsibilities and characteristics of a good teacher of secondary school mathematics are. It seems obvious that these responsibilities and characteristics are different from those of a teacher in the elementary school, or in a college or university. It is not easy to distinguish the special characteristics of a good mathematics teacher from those of any good secondary school teacher. It is easier to analyze what each kind of teacher actually teaches in the way of content.

However, it is not a trivial task to decide what kind of mathematical preparation will make the maximum contribution

to the development of a good mathematics teacher. How much of this development shall we leave to courses in the teaching of mathematics and to student teaching?

After going through this kind of analysis, I believe that a good program of mathematics preparation should satisfy these criteria:

1. mastery of secondary school mathematics characteristic of the years 1963 $+x$
2. breadth of preparation beyond the mathematics of the secondary school
3. experience in learning new mathematics by self-instruction
4. experience with the creative process in minor mathematics research and problem-solving
5. experience in applying high standards of proof to a variety of mathematical problems
6. experience in integrating apparently diverse parts of mathematics
7. experience in relating and applying mathematics to other fields of knowledge

I realize that these criteria are not mutually exclusive, but any extended condensation of them might conceal some of those I consider very important. Further, there are undoubtedly some omissions in the list.

Among the last five criteria are several which seem to have to do more with the teaching and organization of college mathematics than with content per se. The assumptions implicit in this emphasis are: (1) that the task of the teacher is different from that of the future scientist, or engineer, or professional mathematician; (2) that taking a series of courses does not guarantee the attainment of the special competencies needed by the teacher; and (3) that beginning teachers have a tendency to teach mathematics as they have been taught. On the other hand, experiences with self-instruction, creative problem-solving, proof in a variety of contexts, and the interrelating of various parts of mathematics would hardly do serious damage to the preparation of future mathematicians, and teachers of college and university mathematics.

Some justification for the selection of all seven criteria may be helpful. The first hardly needs defense, for how can one teach what he does not know? By mastery I mean far more than the ability to perform. In addition, I mean that the teacher must know under what conditions the concept, or process, or theorem, or formula can be applied. Further, I believe that the teacher must know the genesis of what he teaches, especially when there is a deductive basis for the "what." Does he relate the notion of "multiplicative inverse" to the rule for the division of fractions, and the desire to satisfy postulates, like the distributive principle, to the product of two negative integers?

The second criterion was "breadth of preparation." A teacher cannot fulfill his role as a mathematical guide to his students if he does not know where they have been and where some of them will be in the years to come. He will not have the reservoir of knowledge that contributes to a teacher's confidence and adds to his prestige with his students. Without such breadth he will not be in a position to stimulate and inspire those unusual students who ask questions that call for far more knowledge than what is in today's lesson. To have an abundance of counter-examples on tap, to know, for example, that a straight line is not always the shortest distance between two points, that the whole is not always greater than all of its parts, that a square may contain as many points as a cube, that $x \cdot y$ does not always equal $y \cdot x$, that if the product of two "numbers" is zero neither may be zero, provide a teacher with a perspective that makes him more interesting and his students more interested.

The third criterion dealt with "self-instruction." Since mathematics is constantly growing, and since the mathematics curriculum is likely to be influenced

by this growth, the teacher must be able to learn new mathematics. Breadth of preparation helps in this activity, for the teacher is provided with more established bases as places of departure into new worlds. However, this breadth should be supplemented with experiences of learning some new mathematics while in college, without the aid of professorial lectures and the disciplining effect of periodic class meetings. Such experiences will be of priceless value in keeping up with changes in the mathematics curriculum.

The fourth criterion referred to the "creative process." Poincaré once said " . . . it is by logic we prove, but by intuition that we discover. To know how to criticize is good, but to know how to create is better." His sentiments have been echoed by distinguished mathematicians like Hadamard, Felix Klein, and Pólya. A teacher is only partly educated mathematically if all he can do is perform operations and recall logical proofs. Furthermore, he is poorly prepared to give his students the mathematics education they deserve if he is thus partially developed. A large part of good teaching of mathematics should be characterized by challenges to discover; let the proof come later. If the teacher lacks abundant experiences in discovering, he tends to reflect in his procedures and attitudes his own deficiencies. Mathematics comes to be a matter of getting students to prove relations that other men have dreamed up. Initiative and creativeness are given little exercise; even the driving force that comes from feeling that something is one's own is lost when other men's proposals are the only challenge to investigation.

One does not learn to ski well by always skiing on the same part of the same hill. Likewise, one does not learn how to apply standards of proof to mathematics as a whole by practicing proof in only one part of mathematics. It is well known that even high school students who are taught the art of proof in plane geometry are lost if asked to prove a relation in algebra, or trigonometry, or simple number theory. The preparation of the teacher in mathematics should, for a similar reason, provide adequate experience in proving relations in many parts of mathematics.

The sixth criterion dealt with "integration." It is possible for a prospective teacher of mathematics to take 30 points of mathematics in college and have little opportunity to look for relations and structures that tie some of them into various kinds of unity. These future teachers need an integrating experience with mathematics toward the end of their mathematical work. To look at old things in a new way is a real re-view, not a rehash. The binding relations that are unearthed help to save much of the previous mathematics from the eroding effects of time and disuse.

The seventh criterion had to do with "relating and applying mathematics to other fields of knowledge." I am conscious of the age-old debate on pure and applied knowledge. I am also aware from many years of teaching experience that the learning process of most adolescents requires contact with the world of tangibles, or at least relevancy to other fields of human knowledge. The teacher who relies only on his love and knowledge of mathematics in his teaching in the junior high school, and despises, or is ignorant of, the relation of what he teaches to other areas of human concern, is a "displaced person." Of those who finish junior high school, the per cent who take more than a required freshman course in college is unlikely to be more than 15 per cent. The national movement to have more students take more mathematics is long overdue. However, the teacher of mathematics must be ready to deal with a large number of students whose pursuit of mathematics is more likely to be a strategic retreat than a vigorous attack. Such a teacher must be aware of the applications of mathematics that are likely to be of interest to his students or, at least, have an ample and varied supply of illustrations of the relevancy of mathe-

matics to life outside of the mathematics classroom.

II

The second question I proposed was "how does the present program of mathematical preparation rate with respect to these criteria?" The most common program in the United States consists of about 24 points of mathematics. Included in these are college algebra, plane trigonometry, analytic geometry, or a freshman course that integrates these to some extent. Then follow differential and integral calculus. Next come courses, such as the theory of equations, college geometry, the history of mathematics, the mathematics of finance, and statistics.

A casual glance at this preparation reveals that it will be inadequate in many respects if some of the new curricular proposals are adopted. Some of these seem to call for a knowledge of the algebra of sets, groups, rings, and fields; logic and perhaps some Boolean algebra; functions and relations from a new point of view; an understanding of number bases other than 10; more attention to inequalities as well as equations; some analytic or co-ordinate geometry, involving deductive proof; proofs of number relations; vector analysis as applied to trigonometry; an introduction to the differential and integral calculus of polynomials; probability founded on set notions; and decision theory applied to the discrete binomial distribution.

Despite the fact that a mastery of number systems and the relevant algebras are necessary for the mastery of what is taught in grades 7 to 12, it is not immediately apparent what part of the present program of preparation provides the needed background. More than one research study has revealed this weakness; the findings of Glennon and Orleans[1] are illustrative. The present program, too, hardly provides for breadth in algebra. College algebra and the theory of equations, as usually organized, do not insure that the students will learn much about the structure of algebra, as suggested by the concepts of sets, groups, rings, and fields. The present program provides for little breadth in geometry. Analytic geometry and college geometry hardly touch the foundations of geometry, or the notions of non-Euclidean geometry, or projective geometry, or the idea of topology. Material from advanced calculus or the theory of the real variable might help insure the mastery of the calculus needed in the high school teaching of it. Whether a course in statistics gives the foundations of probability in terms of sets, or the decision theory needed in dealing with the binomial distribution in testing hypotheses, depends on the teacher of the course. These matters should not be left to chance. This course in statistics and the one in the mathematics of finance do provide an opportunity for relating mathematics to other fields, especially social science and business. For applications to the physical sciences at least one course beyond a freshman one in the same field should be taken. Otherwise, the applications must come in the courses in trigonometry, analytic geometry, and the calculus.

How much opportunity is provided for students to learn mathematics on their own is not apparent from a listing of courses. The same comment applies to the provisions for experience in learning what it means to be creative in mathematics. The provision for wide experience in proving mathematical relations seems weak. The proofs in most college algebra and theory of equations textbooks are given in full; the student proves very few relations on his own. The proofs in college geometry are pretty much of the same kind, applied to the same kind of material as is found in high school Euclidean geometry. It is not possible to decide from the names of the courses whether adequate ex-

[1] J. S. Orleans, *The Understanding of Arithmetic Processes Possessed by Teachers of Arithmetic* (New York: College of the City of New York, 1952), and V. J. Glennon and C. W. Hunnicut, *What Does Research Say About Arithmetic?* (Washington, D. C.: Association for Supervision and Curriculum Development, 1952).

perience is given in problem-solving, or whether any study is made of the process with a variety of mathematical illustrations, as is done in Pólya's works. One cannot tell from a course labeled "History of Mathematics" whether the emphasis is on tracing how the race learned mathematics, or how mathematics influenced, and was influenced by, other fields of human study, or whether an attempt was made to pull together or relate the mathematics investigated over the years.

III

The third question was "how should the present program be modified to implement these criteria?" My comments about the extent to which the present program meets these criteria suggest what I think the mathematical preparation of secondary school mathematics teachers should be. What follows is largely recapitulation with some amplification. I will not list the names and sequence of courses, for the organization of these should be left to those who are most competent—the college and university mathematicians. I am quite sure that some new courses will be needed; that some obsolete material should be dropped; that the gaps left by such deletion should be filled by mathematics which is newer and more valuable for teacher preparation. Of course, a large amount of the mathematics in the present preparation of mathematics teachers will be retained. Here are my suggestions:

1. provide for a knowledge of the logical foundations and important properties of the natural numbers, integers, rational numbers, irrational numbers, complex numbers, cardinal numbers, ordinal numbers

   Comment: The natural numbers, integers, and rational numbers might be given such treatment in a freshman course; the next two might be considered in appropriate places in the calculus course.

2. provide for attention to the structure of algebras

   Comment: Much of the material suggested by (1) can be used to illustrate sets, groups, rings, and fields of various kinds. By using modular number systems involving both prime and composite numbers valuable counterexamples can be procured to reveal the special nature of our common number systems. The set language of unions, intersections, and complements should be used whenever it clarifies or simplifies ideas. Sets with an enumerable and nonenumerable number of elements might be used to show how transfinite numbers can be used to contrast real numbers with the rational numbers. The application of sets to the foundations of probability should also be studied.

3. provide for a knowledge of different kinds of geometry

   Comment: Future teachers could profit from knowledge of the foundations of Euclidean geometry; some attention to synthetic non-Euclidean geometry; some study of the basic concepts and principal theorems of synthetic and analytic projective geometry, including groups of transformations; perception of the relation between vectors and matrices; and a knowledge of the elementary notions of differential geometry and combinatorial topology.

4. provide for a background in probability and statistics

   Comment: This background should include at least the knowledge of the set basis of probability; the minimum essentials of "descriptive" statistics; a knowledge of the basic frequency and distribution functions; the meaning of such concepts as sampling and random variable; some study of estimation and hypothesis testing.

5. provide for a command of the calculus and the associated function concept

   Comment: The new calculus textbooks certainly develop the basic concepts more carefully and give the logical

foundations more thoroughly than the older ones. The secondary school mathematics teacher also needs to know more about π and e; how logarithmic and trigonometric tables are constructed; whether the logarithms of negative numbers can be found; how convergence of series is proved; how the areas and volumes of curvilinear configurations are found; whether other kinds of calculus are possible; whether other kinds of trigonometry, based on curves other than the circle, exist; ways of analyzing periodic functions. These needs are only illustrative.

6. provide experience with applications of mathematics to the physical and social sciences
7. provide for a reading knowledge of the history of mathematics
8. provide for experience in integrating mathematics

Comment: Pólya's work[2] in giving a course in mathematics problem-solving is suggestive. Seminars have also been used. The history of mathematics usually has some integrating effect. Other devices are survey courses and courses in foundations.

[2] G. Polya, "On the Curriculum for Prospective High School Teachers," *The American Mathematical Monthly*, LXV, No. 2, 101–4.

Have you read?

GARDNER, MARTIN, "Mathematical Games," *Scientific American*, August 1959, pp. 128–134.

This is one you or your students will not want to miss. What is Phi? Not Pi, but Phi, the Greek letter "ϕ." Perhaps you know that it is an irrational number, it is the golden ratio. That is, if line xy is divided into two segments A and B such that $xy/A = A/B$, then the ratio is the golden ratio of ϕ.

Could you show that Phi equals

1.61803398 . . . ?

What is the reciprocal of Phi? Can Phi be written in other ways, such as

$$\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1}}}}\ \ldots\ ?$$

Did the Pythagorean society use the pentagram because each dimension has a Phi ratio with the next smaller dimension? What happens when you cut squares from a golden rectangle? Did you know that Fibonaccii enters here with his series 1, 1, 2, 3, 5, 8, 13, etc.? The ratio of any two consecutive terms of this series gets closer to Phi as the series increases. There are some interesting paradoxes connected with Phi, which was named after the Greek sculptor Phidias. Put this in the hands of your mathematics majors, and they will be fascinated and elated.—PHILIP PEAK, *Indiana University, Bloomington, Indiana.*

"If a man's wit be wandering, let him study mathematics; for in demonstration, if his wit be called away never so little, he must begin again."
—*Lord Bacon*, Essays: On Studies.

• HISTORICALLY SPEAKING,—

Edited by Howard Eves, University of Maine, Orono, Maine

Notes on the centroid

by Nathan Altshiller Court, University of Oklahoma, Norman, Oklahoma

1

The term "centroid" is of recent coinage. It is used as a substitute for the older term "center of gravity," when the purely geometrical aspects of that point are to be emphasized. The term is peculiar to the English language. The French use "centre de gravité" on all occasions, and others use terms of similar meaning.

The center of gravity, as the name indicates, is a notion that arose in mechanics, most likely in connection with building activities. When, where, and by whom it was invented is not known and probably never will be. It is one of those concepts that occurred to many individuals at many different places, in a more or less confused way.

2

It is possible that during the childhood or even the boyhood days of Archimedes (287–212 B.C.) Euclid was still active in Alexandria. But whether this is so or not, it is certain that when Archimedes himself visited Alexandria, Euclid was no longer there. Thus Archimedes could not have learned much from the author of the *Elements* about the center of gravity, not even the theorem that the medians of a triangle meet in a point—the center of gravity of the triangle—for the simple reason that this proposition is not in Euclid's monumental *Elements*. The first explicit statement of this proposition is due to Heron of Alexandria (perhaps the first century A.D.) and occurs in his *Mechanics*. It may be added, in passing, that the proposition did not become common in the textbooks on plane geometry until the nineteenth century.

While Archimedes does not state that proposition explicitly, the indirect references he makes to it leave no doubt about his familiarity with it. He obtained the centers of gravity of a number of plane figures. Jean Étienne Montucla (1725–1799), the author of the first history of mathematics (1758), declares categorically (vol. I, p. 463) that the center of gravity of solids is a subject Archimedes did not touch.

In 1802 Charles Bossut (1730–1813) published a two-volume *Essai sur l'histoire générale des mathématiques*. This book was highly esteemed by his contemporaries, judging from the fact that within two years after its publication it was already available in translation in Italian (1802–03), English (1803), and German (1804). Bossut credits Archimedes with having found the centroid of plane figures, but has nothing to say about solids.

G. Libri, the author of a four-volume *Histoire des sciences mathématiques en Italie* (Paris, 1838; Halle, 1865), takes sharp issue with Montucla for belittling Archimedes. He finds that there is plenty of evidence, although indirect, that the great Greek mathematician did consider the center of gravity of solids (vol. 3, p. 41, note 2). It would be difficult to decide between Montucla and Libri in their appraisal of the accomplishments of the

Great Syracusan. The preference could only be accorded as a matter of personal feeling, which is another way of saying personal prejudice. Fortunately, we have at present something more convincing to go by. As late as the present century, a hitherto completely unknown work of Archimedes was found in which he derives the center of gravity of a cone, a hemisphere, a segment of a sphere, and a right segment of a paraboloid of revolution (R. C. Archibald [1875–1955], *Outline of the History of Mathematics*, p. 23; published by the Mathematical Association of America, 1949). Libri was right. One will perhaps savor this little story on the history of mathematics more if one bears in mind that Libri is persona non grata with historians of mathematics. Archibald does not mention him, and neither does Cajori nor Bell.

3

Federigo Commandino (1509–1575) proved that the four medians of a tetrahedron meet in a point—the centroid of the solid (*De centro gravitatis solidorum*, 1565). Libri says that the priority right to this discovery was claimed by F. Maurolycus (1494–1575), whom Cajori considers to be the greatest of the sixteenth-century geometers (F. Cajori, *A History of Mathematics*, pp. 141–142, 2nd ed., New York, 1919). According to Libri, both claimants are wrong. His candidate for the honor is that universal genius Leonardo da Vinci (1452–1519). Libri quotes volume and page of the published manuscripts of the great painter to make his point (*ibid.*, p. 41). Libri also quotes this interesting view of Leonardo on mechanics: "La Meccanica e il paradiso delle scienze matematiche" (*ibid.*, vol. 3, p. 40, note 1). Antonina Vallentin, whose biography of Leonardo was received by the critics with great acclaim, testifies to the interest that the Tuscan painter exhibited in the question of the center of gravity (*Leonardo da Vinci*, English translation, p. 206, New York, 1938). On the other hand, E. T. Bell has this to say: "Leonardo da Vinci's published jottings on mathematics are trivial, even puerile, and show no mathematical talent whatever." (*The Development of Mathematics*, 2nd ed., p. 596, note 8, New York, 1945.)

J. L. Coolidge (1873–1954) takes Leonardo da Vinci quite seriously. The famous painter is one of his great amateurs (J. L. Coolidge, *The Mathematics of Great Amateurs*, pp. 43–60, The Clarendon Press, Oxford, 1949). Coolidge agrees with Libri that Leonardo knew the property of the tetrahedron, but he could find no proof of the proposition in the papers of the artist, although he thinks him quite capable of having produced one, and even outlines the probable method that Leonardo might have used.

Coolidge also agrees with the author of the article on Leonardo in the *Enciclopedia Italiana*, vol. 20 (1933), p. 881, who credits the Great Tuscan with having determined the center of gravity of a pyramid of any number of faces, but here again Leonardo does not say how he arrived at the result.

4

The *Enciclopedia* also states that in the case of the triangular pyramid, Leonardo recognized that the three lines joining the midpoints of the three pairs of opposite edges (that is, the bimedians) meet in a point and are bisected by this point, but it is not stated explicitly that Leonardo identified this point with the center of gravity of the tetrahedron, although this is very likely.

In vol. 2, no. 1, January 1809, of the *Correspondance sur l'École Impériale Polytechnique*, edited by Jean Nicolas Pierre Hachette (1769–1834), Gaspar Monge (1746–1818) proves the proposition that the center of gravity of a tetrahedron is at the midpoint of any bimedian of the tetrahedron. Monge gives two demonstrations, both based on dividing the tetrahedron into elements of mass.

In the following issue of the same volume, J. D. Gergonne (1771–1859) re-

minded Monge that Gilles Persone de Roberval (1602–1675) had pointed out that the center of gravity of the mass of a tetrahedron is also the center of gravity of four equal masses placed at the vertices of the tetrahedron, and that this property was incorporated by Jean Louis Lagrange (1736–1813) into his famous *Méchanique Analytique* (1788).

The founder of the first purely mathematical journal, *Annales de Mathématiques* (1810–1832), goes on to show that this property makes Monge's theorem almost obvious. Moreover, it lends itself to prove with ease a number of other properties of the center of gravity that he, Gergonne, had obtained himself, and he reproaches Roberval for having overlooked them, in the following words (my translation): "Truly, one has reason to be surprised that Roberval should have overlooked these various consequences of the observation that he has made; these properties are so simple that one cannot, it seems to me, pass them under silence in the elements of statics, without being guilty of unpardonable negligence."

Whatever one may think of this severe indictment couched in polite language, Roberval's observation made it possible to speak of the center of gravity of four points without associating with the points the mass of the solid of which these points are the vertices. From this there was but one step to extend the same idea to any number of points.

L. N. M. Carnot (1753–1823) associated with n points, coplanar or not, a "focus" whose distance from an arbitrary plane (or axis) is equal to the arithmetic mean of the algebraic distances of the given points from the same plane (or axis). He proposed to call that focus the "center of mean distances" of the given set of n points (*Géométrie de position*, p. 315, Paris, 1803). To prove that the point is unique, Carnot observed that the distances of a point from three planes fix the position of the point in space.

Carnot derived a number of properties of his "center" and called attention to the fact that this point is identical with the center of gravity as considered in mechanics. He drew the conclusion that the center of gravity belongs to pure geometry, and that the mechanical properties of that point should be based on those established by pure geometry (*ibid.*, p. 336).

Carnot's center of mean distances found an enthusiastic supporter in A. L. Crelle (1780–1855), the founder of the *Journal für die reine und angewandte Mathematik* (1826–). In the first volume of his two volume *Lehre der Elemente der Geometrie*, Berlin, 1827, Crelle devotes several pages to the study of the "Mittel-punct der Entfernungen" (pp. 185–192). He concludes with a note in which he supports Carnot's contention that the "Mittelpunct" belongs to pure geometry.

Sylvester's style

by R. A. Rosenbaum, Wesleyan University, Middletown, Connecticut

In most areas of our culture we do not confine ourselves to a unique style. The severity of architecture's International Style seems to be in the process of modification by a New Romanticism; contemporary music pays its respects to traditional melody along with the newer atonality; modern pottery draws inspiration from classical Greece and the Ming Dynasty, as well as from recent sources. But the writers of mathematics don't employ comparably diverse modes: almost all mathematics these days is written in King Tut Style—dry, skeletonized, or at best mummified. As aptly described by Professor Walter Prenowitz, most mathe-

matical prose reads like the end-product of a mathematicizing machine, rather than the creation of a human being.

From one point of view, this situation may be justified. Quite appropriately, mathematics is commonly likened to a language. If the subject is considered to consist merely of the study of the *grammar* of that language (understood in the broadest sense as including the rules of procedure *and* the sentences obtained by use of the rules), the written record should be as condensed and austere as possible. After a list of the undefined terms, the unproved propositions, and the definitions, the author may well begin his (endless?) sequence: "Theorem, proof; Theorem, proof; . . . ." Indeed, some symbolism like Peano's may be chosen to render the exposition independent of any national language—to keep it quite "clean." With such a wrapping, the mummy ought to endure forever.

But *grammar* is only one part of any language. As Dr. Bronowski has pointed out [1],* a knowledge of vocabulary and the ability to *translate* into and from mathematics are indispensable to the mathematician, and also to the layman who wants to know what mathematics is really about. In fact, it is arguable that emphasis on grammar to the exclusion of concern with translation is one source of the common opinion that mathematics (e.g., geometry) is a "dead" language.

Mathematical writing, to be complete, ought to reflect the total professional activity of the mathematician—it should record his efforts at translation as well as his essays in grammar. Only if both elements are present will the exposition be faithful.

There is, of course, an apparently compelling reason for the use of the King Tut Style: publication space is at a premium. The volume of mathematical research is tremendous, and printing costs continue to rise, so that editors must demand ever more condensation. The pace of nineteenth-century writing, reminiscent of a leisurely stroll in the countryside, has given way to the hurry characteristic of high-pressure city life; and the personal, often informal, phraseology of the earlier era has been replaced by a staccato, telegraphic style devoid of humanity, to say nothing of individuality. Comfortable discursiveness, an adequate number of examples, philosophic and aesthetic overtones —all vanish before the editor's inexorable requirement of compression. In the process, *grammar* alone remains; the *translation* which exhibits the origin of the problem, its relation to other problems—its very significance, in fact—is left to the ingenuity of the reader to supply.

In the easier atmosphere of the nineteenth century, no mathematician took greater pains to lay both grammar and translation before his readers than James Joseph Sylvester. A classical scholar who always expressed his thoughts in the most precise words, no matter how many syllables they might have, Sylvester rivals Winston Churchill for the force, the imagery, and the rhythm of his prose. As readers of Bell [2] know, he was a colorful figure, full of vital juices; egocentric, passionate, and prejudiced, yet generous, sympathetic, and understanding. Most of all, he was an *enthusiastic* mathematician, continuously aware of his subject's beauty, and often overwhelmed by it. All this appears clearly in his writing—not just in his Presidential Addresses and his Inaugural Lectures, but in his everyday research papers.

For twenty-five years he had been unsuccessful in his attempt to prove what appeared to be a fundamental theorem of invariants. At the age of sixty-one, Sylvester accepted a professorship at the newly-founded Johns Hopkins University, where he experienced a renascence, and found, along with many other discoveries, the key to his theorem. Fabian Franklin writes [3] of the flush of achievement which accompanied his presentation of the

* Numbers in brackets refer to the references at the end of the article.

proof to the class, and of how the class shared the excitement of great mathematical discovery. Sylvester himself, near the beginning of his paper on the subject, remarks [4]:

> I accomplished with scarcely an effort a task which I had believed lay outside the range of human power,

and a few pages later comments [5]:

> Surely the claim of Mathematics . . . to take a place among the liberal arts must now be admitted as fully made good. Whether we look at the advances made in modern geometry, in modern integral calculus, or in modern algebra, in each of these a free handling of the material employed is now possible, and an almost unlimited scope left to the regulated play of the fancy.

At the end of the paper, Sylvester calls attention to the fact that he found his proof by *enlarging* the scope of the original problem [6]:

> "Divide *et impera*" is as true in algebra as it is in statecraft; but no less true and even more fertile is the maxim "auge *et impera*." The more to do or to prove, the easier the doing or the proof.

The foregoing quotation hints at Sylvester's openness in exhibiting his train of thought, a characteristic explicitly manifest when he writes [7]:

> It was through this idea that I was led to an intuitive perception of the theorems. . . .

Sylvester is well known for the vast number of mathematical terms which he invented. Indeed, he refers to himself as the "Mathematical Adam" [8], and he appends a seven-page glossary of new or unusual terms to a one-hundred-fifty page paper in the *Transactions of the Royal Society* [9]. To a barbarian like myself, some of these terms have the flavor of medical pathology, like "syrrhizoristic" (a disease of the liver?) and "catelecticant" (a mental disturbance?). Others have become standard mathematical terms, like "Hessian," which Sylvester defines, adding the remark [10]:

> named after Dr. Otto Hesse, of Königsberg (the worthy pupil of his illustrious master, Jacobi, but who, to the scandal of the mathematical world, remains still without a Chair in the University which he adorns with his presence and his name) . . .

On another occasion [11], Sylvester epitomizes his attitude toward nomenclature after explaining his choice of the name "discriminant":

> "Discriminant," because it affords the *discrimen* or test for ascertaining whether or not. . . . Progress in these researches is impossible without the aid of clear expression; and the first condition of a good nomenclature is that different things shall be called by different names. The innovations in mathematical language here and elsewhere (not without high sanction) introduced by the author, have been never adopted except under actual experience of the embarrassment arising from the want of them, and will require no vindication to those who have reached that point where the necessity of some such additions becomes felt.

Not all his inventions were given Greek or Latin names. In one of his papers [12] he writes:

> . . . found by the usual rule of "calling" from its conjugate . . . ,

and he attaches a footnote to the word in quotes:

> I borrow this term from the vernacular of the American Stock Exchange.

His remark about what has proved to be one of his most fruitful concepts is particularly interesting [13]:

> It will be convenient to introduce here a notion (which plays a conspicuous part in my new theory of multiple algebra), namely that of the *latent roots* of a matrix—latent in a somewhat similar sense as vapour may be said to be latent in water or smoke in a tabacco-leaf.

Like a number of outstanding mathematicians, Sylvester was a master of several (nonmathematical!) languages, and at one point [14] appealed to his readers as follows:

> I take this opportunity of adding that I shall feel grateful for the communication of any ideas or suggestions relating to this new Calculus from any quarter and in any of the ordinary mediums of language—French, Italian, Latin or German, provided that it be in the Latin character.

He sprinkles Latin terms liberally throughout his papers—not for decoration, but for incisiveness. For example, in his "Instantaneous Proof of Pascal's Theorem," he wishes to call attention to the fact that, with suitable interpretation, he already has a proof of the dual theorem—

there isn't any necessity of "dualizing" a proof. He puts the matter concisely and trenchantly [15]:

. . . the demonstration in the article (applies) *totidem literis* to Brianchon's theorem . . .

Controversies of various sorts have led to bitter invective throughout mathematical history. A catalogue of the insults which mathematicians have hurled at each other would make lively reading. Few such insults can combine the simplicity and force of one addressed by Sylvester to a critic who had missed the point of his argument. This man, says Sylvester [16], has made his attack

with very biting (albeit toothless) criticism.

When the critic humbly acknowledged his mistake, Sylvester sheathed the rapier [17]:

I, on my part, deeply lament the unnecessary tone of acerbity in which my reference to this criticism was couched, and wish I could recall every ungracious expression which it contains.

On most occasions he showed a generous spirit, applauding the discoveries of others. For his friend Cayley, his admiration knew no bounds. After calling attention to an especially significant piece of Cayley's work, Sylvester remarks [18]:

The fortunate proclaimer of a new outlying planet has been justly rewarded by the offer of a baronetcy and a national pension, which the writer of this wishes him long life and health to enjoy. In the meanwhile, what has been done in honour of the discoverer of a new and inexhaustible region of exquisite analysis?

Moved to rapture by the "Salmon-Cayley 27 lines on a cubic surface" and their manifold ramifications, Sylvester gives vent to his feelings [19]:

Surely with as good reason as had Archimedes to have the cylinder, cone and sphere engraved on his tombstone might our distinguished countrymen leave testamentary directions for the cubic eikosi heptagram to be engraved on theirs. Spirit of the Universe! whither are we drifting, and when, where, and how is all this to end?

This list of examples of Sylvester's style, chosen entirely from his research papers, could be vastly extended. But one more quotation will suffice—a quotation which strongly supports the thesis of this article. When Thomas Huxley described mathematics in a way which made it only a "grammar," Sylvester took him to task in his Presidential Address to the British Association, and continued the debate in a footnote to a paper on repeated involutes of circles [20]:

It may be right to mention that the results arrived at in the denumerational portion of this paper have been obtained almost exclusively by processes of observation, comparison, experiment, and induction, satisfying the sense of the perfect and beautiful; and consequently the theory must be regarded as essentially non-mathematical, it having been authoritatively laid down by a great naturalist and eminent public instructor in the *Fortnightly Review* for June 1869, that Mathematics is "that study which knows nothing of observation, nothing of experiment, nothing of induction, nothing" (does any science know much?) "of causation." Lagrange has expressed himself pointedly (in a passage which I cannot at the moment recall) in the very opposite sense to Professor Huxley; so certainly would Fermat, Euler, Gauss, Jacobi, Abel, Cauchy, Eisenstein, Kronecker, Plücker, Riemann, and the other great masters of our art, had their opinion on the subject been appealed to. In a subjective sense this paper owes its origin to my attention having been drawn by the Rev. James White, of Trinity Church, Woolwich, to Capt. Moncrieff's self-reversing gun-carriage, ... The theory has since grown upwards and outwards faster than I have been able to climb after it, like the *beanstalk* of profound significance in the well-known child's story. . . .

Can't we infuse a bit of Sylvester's life into King Tut?

References

1. J. Bronowski, *The Quality of Education* (1947).
2. E. T. Bell, *Men of Mathematics* (1937).
3. J. J. Sylvester, *Mathematical Papers*, vol. IV, p. xxx.
4. *Op. cit.*, vol. III, p. 117.
5. *Op. cit.*, vol. III, p. 123.
6. *Op. cit.*, vol. III, p. 126.
7. *Op. cit.*, vol. III, p. 207.
8. *Op. cit.*, vol. IV, p. 588.
9. *Op. cit.*, vol. I, p. 580.
10. *Op cit.*, vol. I, p. 583.
11. *Op. cit.*, vol. I, p. 280.
12. *Op. cit.*, vol. IV, p. 44.
13. *Op. cit.*, vol. IV, p. 110.
14. *Op. cit.*, vol. I, p. 352.
15. *Op. cit.*, vol. I, p. 151.
16. *Op. cit.*, vol. III, p. 1.
17. *Op. cit.*, vol. III, p. 2.
18. *Op. cit.*, vol. I, p. 203.
19. *Op. cit.*, vol. II, p. 684.
20. *Op. cit.*, vol. II, p. 663.

Reviews and evaluations

Edited by Kenneth B. Henderson, University of Illinois, Urbana, Illinois

BOOKS

Editorial Note: For books of great importance to the teaching profession, THE MATHEMATICS TEACHER has a policy of providing dual reviews written by people who may look at the book from different viewpoints. See *The Growth of Mathematical Ideas, Grades K-12.*

Elements of Modern Mathematics, Kenneth O. May (Reading, Massachusetts: Addison-Wesley Publishing Co., Inc., 1959). Cloth, xvi+607 pp., $6.50.

This introductory textbook for college students cannot be used casually. If it is read carefully, one will find that its author has made an extraordinary effort "to help the student obtain a modest mathematical literacy." In doing this the text represents a trend to emphasize concepts and understandings while *utilizing* modern mathematics. The high school or college instructor interested in exposition which reflects established content in the light of recent mathematical literature and applications of mathematics will find many things of interest in this book.

To actively engage the student with the exposition of the text, lists of questions, called exercises, are included within the text at many key points. These questions deal with ideas which are essential to the development of the text and which are frequently sources of difficulty for students. When a student comes to class or to the instructor for help with a not too uncommon "I don't understand," and the searching for *what* he does not understand begins, the exercises in this text can provide excellent help. To assure that ideas initially grasped through the exposition and exercises are brought into *use*, extensive lists of problems are provided at the end of each section. These problems run from those of a routine nature to the type which could occupy a student for several hours, or weeks, or which, in some cases, could lead him into an area which might challenge him professionally. In addition to the problems which are related specifically to concepts of mathematics, there are many problems with references to their sources in the biological, physical, and social sciences. By selection from the problem lists the instructor should be able to establish the type of student proficiency usually used to establish that learning has occurred.

The development of a concept of proof is initiated in the first chapter on elementary algebra. This chapter begins with material familiar to the student from his study of algebra and geometry in high school. Real numbers are treated as infinite decimals, points on a line, or vectors on a line. Definitions and laws are marked and stated clearly even though the treatment is neither axiomatic nor constructive. The chapter ends with a list of axioms (field), from which the student can deduce propositions previously stated in the chapter. The manner in which topics are treated in this chapter leads naturally into the topics of the next two chapters, elementary logic and set theory. One should not be misled by these chapter titles, as the author cautions in the Preface. Their purpose is to extend and clarify the student's concept of proof and provide tools so that this extension and clarification can continue. To accomplish this, rules of replacement, substitution, inference, hypothesis and choice are presented. These, along with material of symbolic logic and set theory, are intended as a pedagogical device to aid the student. Special notation is adopted in the case of the first two rules, and systematic reference to previously stated or established propositions and to these rules is made throughout the text.

The elaborate nature of the scheme which the text develops with reference to proof may defeat its own purpose. Practicing mathematicians and mathematics instructors proceed along lines elaborated in the discussion of logic but do not find it necessary or even desirable to accompany mathematical exposition with a partially explicated system of logic or notation for logical constants. In spite of this, the material presented is well illustrated and seems within the range of a good college student who will make the effort required by the style of exposition adopted and the variety of symbols employed.

Logic and set theory is used as an organizational device relating mathematics to linguistics, logic, and philosophy. When one thinks of this as an introductory text for a serious student of mathematics and recalls Bertrand Russell's words, "But both [mathematics and logic] have developed in modern times: logic has become more mathematical and mathematics has become more logical," it seems appropriate to call attention to a mathematical formulation of logic and to utilize its symbolism for purposes of rigor and clarity in mathematical contexts where this can be done conveniently. The extent to which this material serves as an effective pedagogical device is an a posteriori question. We are inclined to feel that, in the author's own words, ". . . there remain many unexplored possibilities for further simplifying elementary mathe-

matics" and that they do not require as extensive a treatment of a system or notation of logic as is required to use this text. It cannot be used without emphasis on the rules mentioned above along with the notational device for replacement and substitution and the symbols of logic.

The chapter on elementary set theory establishes a connection between concepts of set theory, elementary algebra, and material of the chapter on logic. Intervals of real numbers and the order of the real numbers constitutes material over which the discussion of sets is built in the sections required for continuity of presentation. This chapter, as well as the chapters which follow on plane analytic geometry, relations and functions, and numbers, tackle each topic from many points of view and *use* the material already presented. In many cases one finds simplifications of presentations available in other texts written for students at this level. The author apparently felt free to use notation and organization which seemed appropriate to the idea being presented. This is an important contribution of this text. Although use of this freedom is not standard practice in elementary texts, it seems that if we want students to develop a feeling for modern mathematics, innovations of this type are needed to help him grasp concepts in a way in which no other approach has helped him. This book makes an effort in this direction in the discussion of the identity function on the reals, relation and function, the converse of a relation, and the arithmetic operations on real-valued functions. To do this many topics which might ordinarily be found in the body of a text are found in the exercises. This seems to hold throughout the text but especially in the last chapters on calculus, probability, statistical inference, and abstract mathematical theories. The last three chapters are considered optional, as are some sections in each chapter. Omitting these and making a careful selection of problems, the remaining material will make a demanding course for most serious students.

This book has many ideas to offer a student beginning his study of college mathematics. It will make demands of his talents and energy, he will probably not find it easy to read, but he should find that his efforts will be rewarded by an awareness of the nature of mathematics and some of its fundamental concepts upon which he can build a continued study of the subject.—*Robert C. Seber, Western Michigan University, Kalamazoo, Michigan.*

The Growth of Mathematical Ideas, Grades K-12, Twenty-fourth Yearbook of the National Council of Teachers of Mathematics (Washington, D.C.: The National Council of Teachers of Mathematics, 1959). Cloth, ix +507 pp., $5.00 ($4.00 to members of NCTM).

Modern mathematics! Will it actually affect our present curriculum if the teachers are not certain what it is, or how it fits into our present-day setting? It was this thought that occurred to the Board of Directors of the National Council of Teachers of Mathematics in 1954 when it proposed that the next yearbook be on in-service education.

The Yearbook Planning Committee states that the book was "not intended to be a textbook on methods, nor a survey of curricula, but rather a highlighting of the most basic mathematical themes which should be central to the entirety of a modern mathematical curriculum." If you examine their product, you will no doubt agree that this book can be used as a textbook of methods for a study group in a school's in-service training program.

The first chapter of this yearbook looks like a mathematics textbook because it presents its objectives in axiomatic and theorem form. This chapter also anticipates the values the teachers of all grade levels can achieve as well as the problems they may encounter in studying the book, for this is not an easy book to digest.

The first theorem, that teachers must plan so that pupils continually have recurring but varied contacts with the fundamental ideas and the processes of mathematics, brings into view not only the importance of having the teachers individually plan recurring experiences for their classes, but also the importance of working together with teachers of other grade levels. Teachers of primary, elementary, junior and senior high schools should understand what confronts each grade-level and work together to bring about recurring experiences.

Primary and elementary teachers may find it difficult to understand some of the more advanced ideas in this book. For this reason, I believe it would be advisable that this book be studied in sessions with teachers of all grade levels. If you must study it alone and the ideas advance beyond your understanding range, I suggest that you read the first twenty pages of chapter 11. It may give you courage to go on.

What are some of the modern ideas which the editors wish to implant? One is that a fraction may be conceived as an ordered pair of integers, e.g., $\frac{2}{5}$ may be (2, 5). On page 42 the editors acquaint you with ordered pairs and with the processes of working with them. Then throughout the succeeding chapters the concept of ordered pairs is used and enlarged.

The discussions on the Cartesian co-ordinates system and on graphing are the clearest I have ever found. This discussion throws light on the understanding of sets. I doubt if any teacher, after a careful reading of this chapter, would not be fairly aware as to what is meant by *set*.

As we know from our teaching experiences, high school students today are frequently not well acquainted with the symbols "$<$ $>$." Understanding these symbols can be helpful in preparing the students for the understanding of other concepts such as the concept of transcendental numbers, functions, etc. An interesting discussion on this very idea is found in this yearbook.

Some mention should be made here of the contributions of the chapter on proof. The na-

ture of probable inference is examined. Is dad (mother) always right? Is seeing believing? (See pp. 114 ff.) This chapter discusses various types of probable inference useful in junior and senior high school and illustrates these with practical examples.

The concepts of validity and consistency of inferences are effectively explained and symbolic logic introduced. In this same chapter Euler's circles are discussed, but Venn diagrams omitted. Venn diagrams are easy to grasp at the junior and senior high grade level and students are attracted to them. Perhaps they should have been included.

Plans of proofs such as counter example, conditional, indirect, *Reductio ad Absurdum*, etc., are differentiated.

Every elementary teacher would profit from studying the chapter on measurement and approximation. It clearly brings to light the errors we frequently teach in the multiplication of decimals, especially in the unit on areas. What approximation of π should be used to give the most accurate answer for the circumference of a circle whose diameter is 68 feet? In chapter 5 (pp. 188 ff.) we find that 3.14 will give as accurate an answer as 3.1416.

Probability is introduced in earlier chapters but a concentrated discussion of it is contained in Chapter 6 with an equally full discussion on statistics in Chapter 7. Chapter 6 discusses activities which develop important concepts of probability, but these activities require the use of dice. The authors give a note of recognition that dice are an aid that is taboo in some communities. Perhaps teachers in some in-service training groups can discuss other aids that develop the ideas of this chapter and share these ideas through THE MATHEMATICS TEACHER.

Both of these chapters lend themselves more to the senior level of high school than to the lower grades, even though a few of the ideas may be introduced earlier. Some of the concepts found in both of these chapters are too difficult for many college students.

The chapter on language and symbolism is most delightful. It not only brings out the importance of certain standard symbolism, but also the importance of an awareness of ideas and concepts that are not yet symbolized. I was surprised, however, that the symbol " $\exists x$" (Existential Quantifier) was not presented in this chapter, since the ideas leading to its presentation were discussed on pp. 363 ff.

Chapter 11, mentioned earlier, deserves special mention because it is a chapter that should be read when the reader of this yearbook gets bogged down and discouraged. No doubt we all realize that one of the greatest difficulties that has confronted us is that we have been on a plateau as far as our mathematics curriculum is concerned. This yearbook can certainly bring on the awareness of our deficiencies. It also gives us a way to approach filling this gap.

The last chapter in the book presents problems that can be used both in the classroom and in group discussions in order to clarify our ideas. These problems are annotated by grade levels for our convenience.—*Olive Leskow, Tolleston School, Gary, Indiana.*

The Twenty-fourth Yearbook is a valuable contribution to mathematics education. It is a "what," "why," and "how" of "modern mathematics"—whatever that term means to you. If it means "modern approach," then Chapter 1, "Number and Operation," Chapter 4, "Proof," Chapter 5, "Measurement and Approximation," and Chapter 8, "Language and Symbolism in Mathematics," fit your definition. If it means "modern content," then Chapter 3, "Relations and Functions," Chapter 6, "Probability," and Chapter 7, "Statistics," are for you. Chapter 1, "The Growth and Development of Mathematical Ideas in Children," Chapter 9, "Mathematical Modes of Thought," and Chapter 11, "Promoting the Continuous Growth of Mathematical Ideas," should be of great help to you. Chapter 10, "Implications of the Psychology of Learning for the Teaching of Mathematics," provides an explanation for it all from a psychological rather than a mathematical viewpoint.

For those who have studied "modern mathematics" in institutes or elsewhere, this book gives the opportunity to see how many newly-acquired concepts "fit together." For those for whom this book will provide a "first exposure," it will prove a very readable, surprisingly complete discourse on many of the topics which have found or may find their way into elementary and secondary mathematics curricula.

A creditable aspect is the consistency of level and spirit in a book with so many authors. Notation and definitions are remarkably uniform.

Adverse remarks should include:

1) There *are* errors—mostly of a minor nature—none which impair the effectiveness of any section.
2) Though approached several times, the confusion involving "fraction," "value of a fraction," "fractional number," "fractional numeral," and "rational number" is not completely removed.
3) The last section of some of the chapters might well have been deleted. In the frustration attending the necessity of expounding a seemingly infinite set of ideas in finitely many pages, too many often find their way into the last section.

If, as I believe, elementary and secondary school teachers *are* interested in the changing thought concerning the curriculum, and if they agree with me that only through an understanding of the underlying content and approach can they make decisions valid for themselves and their students, then this book *should* and *will* be read by all teachers. The Council, the committee and the authors deserve commendation for a worthwhile, readable contribution to the literature of mathematics education.—*Jack E. Forbes, Purdue University, Lafayette, Indiana; Ball State Teachers College, Muncie, Indiana.*

Edited by Joseph N. Payne, University of Michigan, Ann Arbor, Michigan and William C. Lowry, University of Virginia, Charlottesville, Virginia

An introduction to algebra with inequalities

by F. Dawson Trine, Wisconsin State College and Institute of Technology, Platteville, Wisconsin

Much can be gained in algebra by introducing inequalities with equalities. Equations and their solutions are more meaningfully taught in this way, and the student will get a clearer idea of the nature of a mathematical statement. By approaching equations and inequalities through the graphic approach, the student will discover methods of solving them.

In the first few days of a high school algebra course the student is usually introduced to the idea of a variable. In the statement $2x+2=8$, x is the variable. This variable may be replaced by any number the student knows. Some of these numbers make the statement false, and some—one in this case—make it true. If we also include the two statements $2x+2>8$ (read, two x plus two is greater than 8) and $2x+2<8$ (read, two x plus two is less than 8), we find that one, and only one, of these three statements is true for each number that the student knows.

By introducing the number line to the students, a graph for each of these three statements can be drawn. The idea that each integer is represented by a point on the number line can be given. The fractions will take their places between the integers. At this time rational numbers, fractions and integers, plus some irrational numbers such as π and $\sqrt{2}$, will probably be the extent of the student's knowledge. This will be enough at first, but other irrationals may be discussed when it becomes evident that it would be wise, perhaps not until the second year of algebra. In the beginning rational numbers should suffice.

The graphs may be found quite easily. The object is to find which of the three statements is made true by each number tried. If it makes $2x+2>8$ true, it is marked on the number line for this inequality. If the next number tried makes $2x+2<8$ true, it is placed on another number line, in its relative position to the previous number. After a few numbers have been placed on their respective graphs and 3 has been tried and placed on the number line for the equation, it is evident that any number larger than 3 makes $2x+2>8$ true, and any number less than 3 will make $2x+2<8$ true. The practice gained here in substituting values for the variable should prove helpful in evaluating formulas.

The method of graphing is relatively simple. In order to keep matters straight, it is advisable always to write the "greater than" statement at the top and the "less than" statement at the bottom with the statement of equality in the middle. Notice that after each statement a line is drawn to represent a number line. In Figure 1 numbers -1, 0, 1, 2, 3, and 4 have been

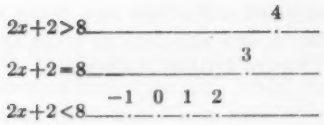


Figure 1

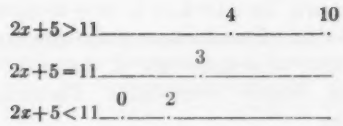


Figure 2

placed on the number line of the statement that each makes true when substituted for the variable.

Many problems of the form $ax+b=c$ should be solved at this time. The students will soon notice that if they find a number that makes the "greater than" statement true and another number that makes the "less than" statement true, the number that will make the "equals" statement true lies somewhere in between. By using this fact and choosing wisely which number to try next, a student can soon graph the three statements, $2x+5>11$, $2x+5=11$, and $2x+5<11$. The easiest and quickest number to substitute for the variable is zero. The substitution of zero makes $2x+5<11$ true. If a table is kept of the values obtained upon substituting each number for the variable, better choices can be made. Table 1 is an example.

TABLE 1

RECORD OF SUBSTITUTIONS IN STATEMENTS OF THE FIRST DEGREE IN THE VARIABLE

	x	$2x+5>11$	$2x+5=11$	$2x+5<11$
(a)	0			$5<11$
(b)	10	$25>11$		
(c)	2			$9<11$
(d)	4	$13>11$		
(e)	3		$11=11$	

From (a) and (b) in Table 1, the number which satisfies the equality is between 0 and 10. From (b) and (c), the solution for the equality is between 2 and 10. Entries (c) and (d) locate it between 2 and 4. As seen in (e), the number 3 makes the equation a true statement. At this stage the graphs are as shown in Figure 2.

We can now tell by inspection that any number greater than 3 will make $2x+5$ greater than 11. Also by inspection it is found that any number less than 3, which includes the negative numbers, will make $2x+5$ less than 11. The completed graphs of the three statements are shown in Figure 3. All points to the right of 3 satisfy the statement $2x+5>11$, and all points to the left of 3 satisfy the statement $2x+5<11$. These answers may be written as they are in the parentheses after each graph. The wavy lines may be used to signify that all these numbers satisfy the statement. The open circle signifies a point not to be included on that part of the graph, i.e., an upper or lower bound.

By graphing such equations as $3x+4+2x=14$, $5x+4=14$, $5x=10$ and $x=2$, each graphed with both inequalities, the student will discover some of the operations that can be performed on equations. Such sets of equations as

$$3(x+2)=15$$

$$3x+6=15$$

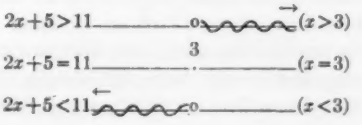
and

$$3(x+2)+2(x-5)=23$$

$$5x-4=23$$

when graphed as above, can be used to introduce operations with parentheses. The

Figure 3



statements $3x+4>4x+7$, $3x+4=4x+7$, $3x+4<4x+7$, and others of the same type may lead to a discovery of the rules for solving simple inequalities. Statements such as

$$\frac{x}{2}+x>6 \qquad \frac{x+2x}{4}>12$$

$$\frac{x}{2}+x=6 \qquad \frac{x+2x}{4}=12$$

$$\frac{x}{2}+x<6 \quad \text{and} \quad \frac{x+2x}{4}<12$$

when graphed as above, lead to methods for solving equations and inequalities with fractions.

The students should be reminded from time to time to look for similarities in problems. They should also learn to inspect the statements to see that the graph does not jump or break again after they have found one number that makes the "equals" statement true. Since we are graphing all three statements and all the numbers are answers, it might be a good idea to refer to a number that separates the numbers which make the "greater than" statement true from those which make the "less than" statement true as a "breaking point." In all of the problems so far discussed the "breaking points" were numbers that make the "equals" statement true. In most of the equations solved in first-year algebra, this will be true. There will, however, be some breaking points which will not make the "equals" statement true. It is suggested that these numbers be placed above the "equals" number line with a box around them to show they do not satisfy the equation. A problem such as

$$\frac{x+2}{x}>2, \qquad \frac{x+2}{x}=2, \qquad \frac{x+2}{x}<2$$

will serve as an example. There is no value for $(x+2)/x$ when x is 0, as division by zero is impossible; thus, zero does not satisfy any one of our three statements. Our graph is discontinuous at this point. Every other number will satisfy one of our three statements, but zero has no place on our graph. Figure 4 shows the graphs.

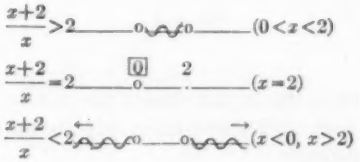


Figure 4

This is a very simple way to introduce discontinuity. There are many interesting problems involving this concept that can be solved. An example would be

$$\frac{x^2+5x+6}{x^2+x-20}>0$$

which is easily solved in the manner described. The concept that division by zero is impossible will certainly be developed while working problems of this type.

Various problems can be solved which will point out the necessity of learning to manipulate algebraic expressions. When the need for manipulation becomes apparent, problems that will give practice should be assigned. The students will see a need for being able to manipulate the expressions readily, and will realize the equations and inequalities are easier to solve when they have gained some facility in manipulation.

The statement $x^2+x-6=0$ is true for two values for the variable, which the students will soon find if they have developed the idea of graphing all three statements. They will also find that if A makes $x^2+x-6<0$ true and B makes $x^2+x-6>0$ true, then between A and B there is a number which will make $x^2+x-6=0$ true. Equations of the form $ax^2+bx+c=0$, which are easily factored, should be given at first and graphed along with their respective "greater than" and "less than" statements. Next, pairs of equations such as

$x^2+5x+6=0$ and $(x+2)(x+3)=0$

should be graphed with their corresponding inequalities. This will bring out the fact that there must be a connection between the two equations since we get the same graph and answers for both. These problems can be solved early in the course and later will lead to the study of multiplying polynomials and factoring quadratics. The graphs are shown in Figure 5.

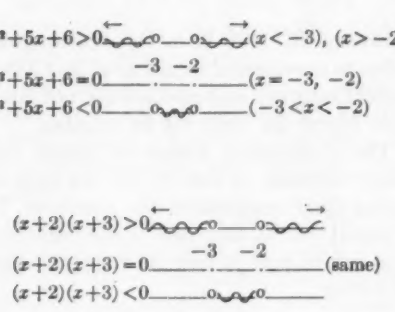


Figure 5

The student can learn much about quadratics by solving one such as $x^2+2x=9$. In this example the student will find one root trapped between 2 and 3. If a table is kept, wise choices can soon trap it between $2\frac{3}{8}$ and $2\frac{1}{4}$, with $2\frac{3}{8}$ being the better choice of the two values. In case a closer approximation to the root is desired the process can continue. The practice with fractions in the substitution process in this problem is very good for developing manipulative skills. When the one root is found near $2\frac{1}{4}$, the other is known to be near $-9\div 2\frac{1}{4}$, (i.e., -4). See Table 2 and notice the symmetry that should be evident to the student after a few problems of this type. Figure 6 shows the accompanying graphs.

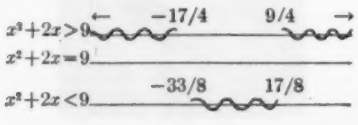


Figure 6

TABLE 2

RECORD OF SUBSTITUTIONS IN STATEMENTS OF THE SECOND DEGREE IN THE VARIABLE

x	$x^2+2x>9$	$x^2+2x=9$	$x^2+2x<9$
0			$0<9$
2			$8<9$
3	$15>9$		
9/4	$9\frac{9}{16}>9$		
$2\frac{1}{4}$			$8\frac{13}{16}<9$
-4			$8<9$
-5	$15>9$		
$-\frac{17}{4}$	$9\frac{9}{16}>9$		
$-4\frac{1}{8}$			$8\frac{47}{64}<9$

After a student has solved a few problems this way, it becomes very easy for him to graph the three statements. As he solves first and second degree equations and inequalities with one variable, he finally realizes that he should first solve the equation and find the "breaking point." Then he can graph the inequalities by checking a number on each side of each breaking point. After more practice he will find that only one number needs to be checked after the breaking points have been found to tell on which number line to start.

The advantages of the method of graphing equalities and inequalities together may be stated as follows: (1) The student gets practice in substituting values for variables and in evaluating expressions. (2) He will be able to solve many equations by guessing and substituting early in the year. (3) He will be able not only to solve equalities, but also inequalities. (4) He may work only with positive numbers in the beginning and extend the approach to include negative numbers after they have been studied. (5) The student will soon realize a need for simple, fast ways to solve equations. (6) He will discover many of the methods of solving equations. (7) The student will see why he must work on certain manipulative skills. (8) He should discover many things by himself and gain a better understanding of algebra.

NCTM

THE NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

Report of the Nominating Committee

The Committee on Nominations and Elections presents its slate of nominees for offices to be filled in the 1960 election. The term of office for the president and the two vice-presidents is two years. Three directors are to be elected for terms of three years.

In making nominations for the three director positions, the Committee followed the directive adopted by the Board of Directors in 1955 which states, "Nominations shall be made so that there shall be not more than one director elected from each state, and that there shall be one director, and not more than two, elected from each region." Members may consult THE MATHEMATICS TEACHER for October, 1955, for a map of the regions as they are now defined.

Ballots will be mailed on or before February 26, 1960 from the Washington Office to members of record as of that date. Ballots returned and postmarked not later than March 26, 1960 will be counted.

The Committee wishes to thank the many members of the NCTM for help in giving their suggestions for nominees. It is hoped that all members of our organization will be sure to exercise their privilege of voting.

IDA BERNHARD PUETT, *Chairman*
MILTON BECKMANN
CHARLES BUTLER
HOWARD F. FEHR
KENNETH B. HENDERSON
HOUSTON T. KARNES
ALBERT LINTON
ANN PETERS
OSCAR SCHAAF

NOMINEES FOR PRESIDENT

PHILLIP S. JONES

HENRY VAN ENGEN

Phillip S. Jones

Professor of Mathematics and Education, University of Michigan, Ann Arbor.

A.B., M.A., Ph.D., University of Michigan.

Jackson High School, Jackson Junior College, Jackson, Mich.; Edison Institute of Technology, Dearborn, Mich.; University School, Ohio State University; Visiting Associate Professor, University of California; instr., Summer Inst., Duke Univ.

Member: NCTM; AMS; MAA; Phi Rho Pi; Sigma Xi; Phi Kappa Phi; AAAS, History of Science Society.

Activities in NCTM: Vice-President; member, Board of Directors; Departmental Editor, Associate Editor, THE MATHEMATICS TEACHER; Editor, *24th Yearbook*.

Other activities: member, NAS-NRC Film Committee; SMSG Advisory Committee; Mathematics Committee, CEEB; formerly member of Board of Governors of MAA; Co-operative Committee on the Teaching of Science and Mathematics of the AAAS.

Publications: Articles in THE MATHEMATICS TEACHER, *The American Mathematical Monthly*, and other magazines.

Henry Van Engen

Professor of Education and Mathematics, University of Wisconsin, Madison, Wisconsin.

A.B., Nebraska Wesleyan University; Ph.D., University of Michigan.

Elementary, junior high, and senior high schools, Nebraska, Michigan, and Ohio; Western Reserve University; Kansas State University; Iowa State Teachers College.

Member: NCTM; MAA; AMS; Sigma Xi; Phi Kappa Phi; Phi Beta Kappa; Pi Mu Epsilon; Kappa Mu Epsilon.

Activities in NCTM: Editor, THE MATHEMATICS TEACHER; member, Board of NCTM; member, Elementary Curriculum Committee.

Other activities: Past President, Kappa Mu Epsilon; member, SMSG Advisory Committee, Commission on Mathematics, CEEB.

Publications: Numerous articles appearing in THE MATHEMATICS TEACHER, *School and Society*, *School Science and Mathematics*, *The Arithmetic Teacher*, and other magazines; chapters in various NCTM yearbooks; coauthor of textbooks.

NOMINEES FOR VICE-PRESIDENT—SECONDARY SCHOOL LEVEL

WILLIAM H. GLENN

EUNICE LEWIS

William H. Glenn

Mathematics Supervisor, Pasadena City Schools, Pasadena, California.

A.B., M.A., University of California at Los Angeles; further graduate work at California Institute of Technology, U.C.L.A., and Los Angeles State College.

Mathematics teacher, Pasadena Junior College, 1939–43; physics teacher, Army Specialized Training Program, 1943–44; assistant director, Research and Development Laboratory, A. O. Smith Corporation, Pacific Coast Division, 1944–46; chairman, Natural Science Division, John Muir College, 1946–50; assistant coordinator, Natural Science Division, Pasadena City Schools, 1950–53; Chairman, Mathematics and Astronomy Department, Pasadena City College, 1953–54; Assistant Principal, John Muir High School, Pasadena, 1954–56; Mathematics Supervisor, Pasadena City Schools, 1956–.

Study group leader three different summers at the California Conference for Teachers of Mathematics and member of the planning committee for all nine of the sessions; co-chairman of an arithmetic workshop in the summer of 1957 in Pasadena; chairman of two Pasadena workshops in the summer of 1959, one in elementary arithmetic and one in secondary mathematics; instructor in content and methods class in arithmetic for Pasadena teachers, 1959.

Member: NCTM; NEA; MAA; ASEE; ASCD; ACEI; PDK; CASMT; AVEAC; CTA; CMC; PEA.

Activities in NCTM: California State Representative; member of NCTM Committee on Co-operation with Industry; Associate Editor, *Mathematics Student Journal;* speaker and discussion leader at many NCTM conventions.

Other activities: Secretary, California Mathematics Council, Southern Section, 1951–53; President, California Mathematics Council, 1953–55; Treasurer, California Science Teachers Association, Southern Section, 1951–53; President, Beta Delta Field Chapter, Phi Delta Kappa, 1954; Vice-President, Pasadena Education Association, 1952.

Publications: Articles in the *Twenty-second Yearbook of NCTM*, THE MATHEMATICS TEACHER, *The Arithmetic Teacher, Teaching Tools, The California Mathematics Council Bulletin, The Clearing House,* and *The Bulletin of the National Association of Secondary School Principals.* Mathematics consultant for *Fractions Kit,* Models of Industry. Co-developer of *The Direct-O-Percenter.* Co-ordinator of mathematics publications for the Pasadena City Schools: *Tips to Teachers; General Mathematics Guide; Using the Language of Algebra;* and *Exploratory Algebra.*

Eunice Lewis

Associate Professor at the University of Oklahoma; teacher of mathematics at the University High School and supervisor of mathematics teacher trainees.

B.A. in mathematics, University of Oklahoma; M.A. in mathematics, University of Oklahoma.

Teacher of mathematics, Covington High School, Covington, Oklahoma; Sapulpa High School, Sapulpa, Oklahoma; Tulsa Central High School, Tulsa, Oklahoma; University High School, University of Oklahoma, Norman, Oklahoma.

Activities in NCTM: Oklahoma State Representative for the past 15 years; Southwest Regional Chairman for the affiliated groups.

Member: Oklahoma Council of Teachers of Mathematics, Mathematical Association of America, Oklahoma Education Association, National Education Association, Delta Kappa Gamma, Kappa Delta Pi, Pi Mu Epsilon.

Other activities: Appeared on programs of the National Council of Teachers of Mathematics, and others of national, state, and local scope. Consultant at the Mathematics Institute held at Louisiana State University for three summers; at the Mathematics Institute held at the

University of Oklahoma for two years; at the Mathematics Conference held at Central State College, Edmond, Oklahoma; and at in-service conferences sponsored by schools in Oklahoma. A visiting lecturer at the Mathematics and Science Institute sponsored by the National Science Foundation at the University of Oklahoma, during the summer of 1959; a visiting lecturer at the University of Illinois, 1958–1959, to work with the University of Illinois committee on School Mathematics and with the Arithmetic Project. Member of the School Mathematics Study Group held at Yale during the summer of 1958 and at the University of Colorado during the summer of 1959.

Publications: Several articles published in THE MATHEMATICS TEACHER and other periodicals.

NOMINEES FOR VICE-PRESIDENT—ELEMENTARY SCHOOL LEVEL

JOYCE BENBROOK

CLARENCE ETHEL HARDGROVE

Joyce Benbrook

Professor of Elementary Education, University of Houston, Houston, Texas.

M.A., Ph.D., The University of Texas.

Elementary and junior high school teacher and elementary school principal, Rosenberg, Texas; arithmetic teacher, grades 7 and 8, Teague, Texas; elementary teacher, Houston, Texas; Teaching Fellow, University of Texas.

Member: NCTM; Texas State Teachers Association; Texas Elementary Principals and Supervisors Association; Texas Council of Teachers of Mathematics; Houston Council of Teachers of Mathematics; Phi Beta Kappa; Pi Sigma Alpha; Pi Lambda Theta; Phi Kappa Phi; Kappa Delta Pi; Delta Kappa Gamma.

Activities in NCTM: Member, Elementary Curriculum Committee, 1955–59, Handbook Committee, 1953–55.

Other activities: Member of the organization committee and first President of the Texas Council of Teachers of Mathematics; Chairman, Elementary Curriculum Committee of the Mathematics Study Commission of the Texas Education Agency, 1958–59; Director of the Mathematics Institutes, University of Houston, 1951–55; member of the Louisiana State University Mathematics Institute staff, 1954; member of Baylor University Arithmetic Workshop staff, 1952; consultant in arithmetic for several Texas school systems; speaker at local, district, and state meetings of mathematics teachers; participant, National Science Foundation Institute on the Teaching of Arithmetic, University of Michigan, 1959.

Publications: Coauthor of *Working with Numbers*, worktexts for grades 1, 2, and 3, *Contributions in Arithmetic*, and *Working with Numbers Teaching Aids*.

Clarence Ethel Hardgrove

Professor of Mathematics, Northern Illinois University, DeKalb, Illinois.

A.B., Marshall College, Huntington, W. Va.; M.A., West Virginia University; Ph.D., Ohio State University.

Teacher in the elementary, junior high, and secondary schools of West Virginia; instructor, Ohio State Univ.; Professor of Math., Northern Illinois Univ.

Member: National Council of Teachers of Mathematics; Illinois Council of Teachers of Mathematics; Ohio Council of Teachers of Mathematics; Central Association of Science and Mathematics Teachers; Association for Supervision and Curriculum Development; Association of Childhood Education; Kappa Delta Pi; Pi Lambda Theta; Sigma Zeta; AAUW; AAUP.

Activities: Vice-President, Illinois Council of Teachers of Mathematics; Chairman of Elementary Section, Central Association of Science and Mathematics Teachers; speaker at numerous mathematics meetings of National and Illinois Councils of Teachers of Mathematics and other groups; consultant for curriculum committees, workshops, and institutes; chairman, Illinois Curriculum Program Committee on Mathematics Curriculum of the Elementary School.

Publications: Coauthor, *Thinking in the Language of Mathematics*, Illinois Curriculum Program; articles in *Twenty-second Yearbook* and *Twenty-fifth Yearbook* of the National Council of Teachers of Mathematics; articles in periodicals; Bulletins for Northern Illinois University Bulletin Service.

NOMINEES FOR THE BOARD OF DIRECTORS

J. Houston Banks

Irvin H. Brune

Eva A. Crangle

Z. L. Loflin

Eugene D. Nichols

Robert E. K. Rourke

J. Houston Banks

Southeastern Region

Professor of Mathematics, George Peabody College for Teachers.

B.S., Tennessee Polytechnic Institute; M.A. and Ph.D., George Peabody College for Teachers.

Teacher and principal, elementary schools and high schools; Dean and mathematics instructor, East Central Junior College, Decatur, Mississippi; Mathematics Instructor, 133rd Detachment, Army Air Corps; Professor of Mathematics, Florence State College, Alabama; Visiting Professor of Education, Alabama Polytechnic Institute.

Member: NCTM; MAA; NEA; Tennessee Teachers of Mathematics; Tennessee Academy of Science; TEA; Phi Delta Kappa; Kappa Delta Pi; Pi Mu Epsilon.

Activities in NCTM: Member, Committee on Supplementary Publications; Southeastern Representative for Affiliated Groups; Yearbook Planning Committee; former State Representative.

Other activities: Speaker at NCTM and other professional meetings; consultant assignments; guest lecturer, institutes, workshops and conferences; Chief Consultant in Mathematics, statewide N.S.F. summer high school program for the gifted; active in organizing Tennessee Teachers of Mathematics, President 1954–55; active part in organizing and conducting state-wide high school mathematics contests; chairman, committee on high school contests, Southeastern Section, MAA; former President, Mathematics Section, TEA; former State Representative, NCTM. Research in relative effectiveness of various types of college freshman programs, in best utilization of graduate assistants, and in teaching by television.

Publications: Contributor to THE MATHEMATICS TEACHER, *Tennessee Academy of Science Journal;* author, *Mathematics for General Education, Elements of Mathematics, Learning and Teaching Arithmetic.*

Irvin H. Brune

North Central Region

Professor of Mathematics, Iowa State Teachers College, Cedar Falls, Iowa.

B.S., The College of Wooster; M.A., Ph.D., Ohio State University.

Senior high school teacher and principal, Lexington, Ohio; junior and senior high schools, Mansfield, Ohio; Montclair, New Jersey, State Teachers College; Frostburg, Maryland, State Teachers College.

Member: NCTM; MAA; CASMT; AAUP; Iowa Association of Mathematics Teachers; Iowa Academy of Science; Phi Beta Kappa; Phi Delta Kappa; Alpha Delta Omega; Kappa Mu Epsilon.

Activities in NCTM: Assistant Editor, THE MATHEMATICS TEACHER, 1953–59; Editor, *Revised Guidance Pamphlet,* 1953; Chairman, Committee on Guidance Pamphlet, 1959. Member, Conference on Directions, 1959.

Other activities: Advisory Board, IAMT; Chairman, Committee on High School-College Teaching, Iowa Section, MAA.

Publications: articles in THE MATHEMATICS TEACHER, *The Arithmetic Teacher, NEA Journal, Education, Midland Schools, Bulletin of General Semantics, Bulletin of the National Association of Secondary School Principals;* chapter in the *Twenty-first Yearbook,* NCTM.

Eva A. Crangle

Western Region

Teacher-consultant in Mathematics, Junior and Senior High Schools, Salt Lake City Public Schools.

B.S., University of Nebraska; M.A., Columbia University; Shell Merit Fellowship, Leland Stanford University, summer 1956; NSF In-service Institute, University of Utah, 1959–60.

Junior high school mathematics teacher, Bryant Junior High, Salt Lake City;

senior high school mathematics teacher, West High and Highland High, Salt Lake City, Utah.

Member: NCTM; NEA; Utah Council of Teachers of Mathematics; Utah Education Association; Salt Lake City Principals and Supervisors Association; Salt Lake City Teachers Association; Pi Lambda Theta; Kappa Delta Pi; Delta Kappa Gamma.

Activities in NCTM: present State Representative for NCTM in Utah; chairman of Local Arrangements for Twentieth Summer Meeting, 1960; participant on NCTM Convention programs; committee member; convention delegate representing Utah.

Other activities: Member citywide textbook committees; member evaluation team, Kellogg Evaluation Study (under the auspices of the Kellogg Foundation); member Utah State Mathematics Curriculum Committee; participant in annual programs of the Mathematics Section, Utah Education Association conventions.

Publications: Contributor to the Utah Mathematics Course of Study, Grades Seven to Twelve; unpublished supervisory bulletins and outlines for summer school course of study.

Eugene D. Nichols

Southeastern Region

Associate Professor of Mathematics Education, Florida State University, Tallahassee, Florida.

B.S., University of Chicago; M.A. and Ph.D., University of Illinois.

Teacher of Mathematics at University School, University of Illinois; Urbana High School, Urbana, Illinois; Illinois Commercial College, Champaign, Illinois; instructor in Mathematics, Roberts Wesleyan College, N. Chili, New York; Director, 1958 and 1959 NSF Sponsored Summer Mathematics Programs for Talented High School Students.

Member: NCTM; MAA; AMS; Illinois CTM; Florida CTM; Florida Education Association; Phi Delta Kappa; Kappa Delta Pi; Pi Mu Epsilon.

Activities in NCTM: Appearances on convention programs.

Other activities: member, MAA Committee on High School Contests; Florida Director, MAA High School Contest; member, Curriculum Committee, Florida CTM; Editor, *Florida Math Journal;* Past President, Illinois Alpha Chapter of Kappa Delta Pi; consultant to mathematics teacher institutes; mathematics editor for a textbook publisher.

Publications: Coauthor of a chapter in NCTM 24th Yearbook, articles in THE MATHEMATICS TEACHER, *The Mathematics Student Journal, Bulletin of the National Association of Secondary School Principals, The Social Studies.*

Z. L. Loflin

Southwestern Region

Professor and Head, Mathematics Department, Southwestern Louisiana Institute.

B.S., M.S., Louisiana State University; Ph.D., Columbia University.

Member: NCTM; Phi Kappa Phi; Pi Mu Epsilon; Kappa Mu Epsilon; Louisiana Teachers Association; NEA; American Educational Research Association; American Mathematical Association; American Chemical Society; AAUP; American Institute of Mining and Metallurgical Engineers; American Association for Engineering Education; Central Association of Science and Mathematics Teachers; Rotary Club of Lafayette. Listed in *American Men of Science* and *Who's Who in the South and Southwest.*

Activities in NCTM: Editorial Board, THE MATHEMATICS TEACHER.

Other activities: member, National Board of Governors of Mathematical Association of America; past national President, Theta Xi.

Robert E. K. Rourke

Northeastern Region

Head, Department of Mathematics, Kent School, Kent, Connecticut.

Honour B.A., medallist in mathematics, Queen's University, Kingston, Ontario; A.M., Shattuck Scholar, Harvard; course work for doctorate in mathematics, Harvard.

Teacher, Head, Department of Mathematics, Headmaster, Pickering College, Newmarket, Ontario.

Summer School lecturer at Queen's University; University of Alberta; Teachers College, Columbia University. Consultant on mathematics to numerous school systems.

Activities: Past President, Ontario Association of Teachers of Mathematics and Physics; formerly on executive of Canadian Mathematical Congress; director of NCTM 1957–59; Executive Director of the Commission on Mathematics of the CEEB; member of Advisory Council, School Mathematics Study Group; committee of MAA re Survey and Seminar of OEEC.

Publications: Coauthor of *An Advanced Course in Algebra; Trigonometry and Statics; Mathematics for Canadians*, Books I, II, III; articles in professional literature.

Your professional dates

The information below gives the name, date, and place of meeting with the name and address of the person to whom you may write for further information. For information about other meetings, see the previous issues of THE MATHEMATICS TEACHER. Announcements for this column should be sent at least ten weeks early to the Executive Secretary, National Council of Teachers of Mathematics, 1201 Sixteenth Street, N.W., Washington 6, D. C.

NCTM convention dates

JOINT MEETING WITH MATHEMATICAL ASSOCIATION OF AMERICA

January 30, 1960
Conrad Hilton Hotel, Chicago, Illinois
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

THIRTY-EIGHTH ANNUAL MEETING

April 20–23, 1960
Statler-Hilton Hotel, Buffalo, New York
Louis F. Scholl, Board of Education, Buffalo 2, New York

JOINT MEETING WITH NEA

June 29, 1960
Los Angeles, California
M. H. Ahrendt, 1201 Sixteenth Street, N.W., Washington 6, D. C.

TWENTIETH SUMMER MEETING

August 21–24, 1960
University of Utah, Salt Lake City, Utah
Eva A. Crangle, Board of Education, Salt Lake City 11, Utah

Other professional dates

Georgia Mid-Winter Mathematics Conference

February 5–6, 1960
Rock Eagle 4-H Center, Eatonton, Georgia
James P. Brown, Southwest High School, 3116 Sewell Road, S.W. Atlanta 11, Georgia

Mathematics Section, New York Society for the Experimental Study of Education

February 13, 1960
1056 Thompson Hall, Teachers College, Columbia University, New York, New York
John A. Schumaker, Secretary, Montclair State College, Upper Montclair, New Jersey

Association of Teachers of Mathematics of New York City

February 27, 1960
Room 846, Washington Irving High School, 40 Irving Place, New York, New York
Lee Batch, New Utrecht High School, 17th Avenue and 80th Street, Brooklyn, New York

Program for joint meeting of MAA and NCTM

The following program has been announced for the joint sessions of the MAA and NCTM to be held in the Eighth Street Theater, Chicago, Illinois, on January 30.

GENERAL SESSION, 9:00 to 10:15 A.M.

Presiding: Carl B. Allendoerfer, University of Washington, Seattle, Washington.

Speakers:

Paul C. Rosenbloom, University of Minnesota, Minneapolis, Minnesota—"Short and Long Range Improvement of the School Mathematics Curriculum."

Morris Kline, New York University, New York, N. Y.—"Better Curriculum or Better Pedagogy?"

SESSION ON ADVANCED PLACEMENT, 10:30 to 11:45 A.M.

Moderator: Harold P. Fawcett, Ohio State University, Columbus, Ohio.

Collegiate Views and Experiences: E. P. Vance, Oberlin College, Oberlin, Ohio; W. T. Scott, Northwestern University, Evanston, Illinois.

High School Views and Experiences: Edwin C. Douglas, The Taft School, Watertown, Connecticut; Henry Swain, New Trier Township High School, Winnetka, Illinois.

SESSION ON THE NATURE AND ROLE OF GEOMETRY IN HIGH SCHOOL, 2:00 to 3:15 P.M.

Moderator: Max Beberman, University of Illinois, Urbana, Illinois.

Collegiate Views: Kenneth B. Leisenring, University of Michigan, Ann Arbor, Michigan; Robert R. Christian, University of British Columbia and University of Illinois, Urbana, Illinois.

High School Views: Henry Syer, Kent School, Kent, Connecticut; Roderick McLennon, Arlington Heights High School, Arlington Heights, Illinois.

SESSION ON CALCULUS IN HIGH SCHOOL, 3:15 to 4:30 P.M.

Moderator: Philip S. Jones, University of Michigan, Ann Arbor, Michigan.

Collegiate Views: Albert A. Blank, New York University, New York, N. Y.; J. H. Neelley, Carnegie Institute of Technology, Pittsburgh, Pennsylvania.

High School Views: W. Eugene Ferguson, Newton High School, Newtonville, Massachusetts; Hubert Davis, Cranbrook School, Bloomfield Hills, Michigan.

Contest in High School Mathematics

The Eleventh Annual Contest in High School Mathematics, which is sponsored by the Mathematical Association of America and the Society of Actuaries, will be held on March 10, 1960.

Invitations have been sent to high schools in the United States and Canada. If your high school did not receive an invitation and a registration form, please request them from Dr. W. H. Fagerstrom, Contest Director, Pan American College, Edinburg, Texas.

National Science Foundation Academic Year Institutes

Approximately 1,600 experienced teachers will be enrolled in National Science Foundation Academic Year Institutes at 33 Colleges and universities in 1960–61. Most will be secondary school teachers; a limited number of openings for college teachers will also be available in some of the Institutes. Each teacher will pursue a a program of study in science or mathematics planned especially for him and conducted by scientists noted both for competence in their fields and for skill in presentation.

The Foundation grants will provide stipends of $3,000 for each participant, with additional allowances for dependents, books, and travel. Some Institutes will provide an additional summer training program to enable teachers more easily to fulfill graduate degree requirements. Supplementary allowances will be provided for teachers participating in this extended program.

Listed below are the institutions receiving grants, the names of Institute directors, and the subject fields of each Institute. Information and application forms can be obtained from the directors of the individual Institutes, *not* from the National Science Foundation.

ALABAMA

Tuskegee Institute, Tuskegee Institute, Alabama
Dr. W. Edward Belton, Department of Chemistry
(Biology, Chemistry, Mathematics)

CALIFORNIA

San Diego State College, San Diego 15, California
Dr. John E. Eagle, Department of Mathematics (Mathematics)

Stanford University, Stanford, California
Prof. Harold M. Bacon, Department of Mathematics
(Chemistry, Mathematics)

COLORADO

University of Colorado, Boulder, Colorado
Prof. John M. Cleveland, Department of Physics
(Biology, Chemistry, Mathematics, Physics)

GEORGIA

Atlanta University, Atlanta 14, Georgia
Dr. K. A. Huggins, Department of Chemistry
(Biology, Chemistry, Mathematics)

University of Georgia, Athens, Georgia
Dr. Jonathan J. Westfall, Head, Department of Botany
(Biology, Chemistry, Mathematics, Physics)

ILLINOIS

University of Illinois, Urbana, Illinois
Prof. Joseph Landin, Department of Mathematics
(Mathematics)

INDIANA

University of Notre Dame, Notre Dame, Indiana
Prof. Arnold E. Ross, Head, Department of Mathematics
(Mathematics)

IOWA

Iowa State Teachers College, Cedar Falls, Iowa
Dr. Robert A. Rogers, Department of Science
(Biology, Chemistry, Earth Sciences, General Science, Mathematics, Physics)

KANSAS

University of Kansas, Lawrence, Kansas
Prof. Lee M. Sonneborn, Department of Mathematics
(Mathematics)

LOUISIANA

Louisiana State University, Baton Rouge 3, Louisiana
Dr. Houston T. Karnes, Department of Mathematics
(Mathematics)

MASSACHUSETTS

Boston College, Chestnut Hill 67, Massachusetts
Rev. Stanley J. Bezuszka, S. J., Chairman, Dept. of Mathematics
(Mathematics)

Harvard University, Cambridge 38, Massachusetts
Dean Francis Keppel, Graduate School of Education
(Biology, Chemistry, Mathematics, Physics)

MICHIGAN

Michigan State University, East Lansing, Michigan
Dr. John Mason, Science and Mathematics Teaching Center
(General Science)

University of Michigan, Ann Arbor, Michigan
Dr. Leigh C. Anderson, Department of Chemistry
(Chemistry, Mathematics, Physics)

MINNESOTA

University of Minnesota, Minneapolis 14, Minnesota
Prof. Charles Hatfield, Department of Mathematics
(Mathematics)

MISSOURI

Washington University, St. Louis 30, Missouri
Prof. E. U. Condon, Chairman, Department of Physics
(Astronomy, Biology, Chemistry, Earth Sciences, General Science, Mathematics, Physics)

NEW MEXICO

New Mexico Highlands University, Las Vegas, New Mexico
Dr. E. Gerald Meyer, Department of Chemistry
(Biology, Chemistry, Mathematics, Physics)

NEW YORK

Syracuse University, Syracuse 10, New York
Prof. Alfred T. Collette, Department of Bacteriology and Botany
(Astronomy, Biology, Chemistry, Earth Sciences, Mathematics, Physics)

NORTH CAROLINA

University of North Carolina, Chapel Hill, North Carolina
Dr. Edwin C. Markham, Department of Chemistry
(Biology, Chemistry, Earth Sciences, Mathematics, Physics)

OHIO

Ohio State University, Columbus 10, Ohio
Dr. John S. Richardson, Department of Science Education
(Biology, Chemistry, Mathematics, Physics)

OKLAHOMA

Oklahoma State University, Stillwater, Oklahoma
Dr. James H. Zant, Department of Mathematics
(Biology, Chemistry, Earth Sciences, Mathematics, Physics)

OREGON

Oregon State College, Corvallis, Oregon
Dr. Stanley E. Williamson, Chairman, Department of Science Education
(Biology, Chemistry, Earth Sciences, Mathematics, Physics)

PENNSYLVANIA

Pennsylvania State University, University Park, Pennsylvania
Prof. William H. Powers, Department of Chemistry
(Biology, Chemistry, Earth Sciences, Mathematics, Physics)

University of Pennsylvania, Philadelphia 4, Pennsylvania
Dean William E. Arnold, School of Education
(Biology, Chemistry, Mathematics, Physics)

PUERTO RICO

University of Puerto Rico, Rio Piedras, Puerto Rico
Dean Mariano Garcia, Department of Mathematics
(Mathematics)

RHODE ISLAND

Brown University, Province 12, Rhode Island
Prof. Elmer R. Smith, Chairman, Department of Education
(Biology, Chemistry, Mathematics, Physics)

SOUTH DAKOTA

State University of South Dakota, Vermillion, South Dakota
Prof. Charles M. Vaughn, Head, Department of Zoology
(Astronomy, Biology, Chemistry, Geology, Mathematics, Physics)

TEXAS

University of Texas, Austin 12, Texas
Dr. Robbin C. Anderson, Department of Chemistry
(Biology, Chemistry, Earth Sciences, Mathematics, Physics)

UTAH

Brigham Young University, Provo, Utah
Dr. Lane A. Compton, Department of Physics
(Chemistry, Mathematics, Physics)

University of Utah, Salt Lake City 12, Utah
Dr. Thomas J. Parmley, Head, Department of Physics
(Biology, Chemistry, Mathematics, Physics)

VIRGINIA

University of Virginia, Charlottesville, Virginia
Dr. James W. Cole, Jr., Department of Chemistry
(Biology, Chemistry, Earth Sciences, General Science, Mathematics, Physics)

WISCONSIN

University of Wisconsin, Madison 6, Wisconsin
Dr. Donald H. Bucklin, Department of Zoology
(Biology, Chemistry, Mathematics, Physics)

Wiley BOOKS

BASIC CONCEPTS of ELEMENTARY MATHEMATICS

By WILLIAM L. SCHAAF, *Brooklyn College*

This new text is designed primarily for "elementary education majors" in teacher-training institutions and liberal arts colleges.

Its primary purpose is to afford the reader an opportunity to gain some insight into such fundamental ideas as—

(1) abstract nature and logical structure of "pure" arithmetic
(2) relation of arithmetic to algebra, geometry, trigonometry and analysis
(3) relation of all mathematics to logic and axiomatics

Significant features:

- Contains over 600 examples, problems and questions
- Places exercises within exposition, not at the ends of chapters
- Treats measurement and mensuration; finance and insurance
- Indicates new trends in mathematics curriculum

Ready February 1960. Approx. 380 pages. Prob. $4.95

Reserve your examination copy now.

JOHN WILEY & SONS, Inc. 440-4th Ave., New York 16, N.Y.

Just published . . .

HIGH SCHOOL GEOMETRY Keniston, Tully

This new geometry text will meet all your requirements for a course reflecting modern trends. Suitable for either college preparatory or general education classes.

Includes considerable solid geometry—introduces coordinate geometry—uses the concept of a set in work with locus and definitions.

Effectively maintains basic skills of algebra and arithmetic.

Numerous illustrations and diagrams, with second color used functionally.

Teachers' Manual and Answer Book available. Write for full information.

Sales Offices:
NEW YORK 11
CHICAGO 6
ATLANTA 3
DALLAS 1
PALO ALTO
TORONTO 16

Home Office:
BOSTON

GINN AND COMPANY

YOU KNOW that

there are new things coming
in school mathematics . . .

The SYRACUSE UNIVERSITY

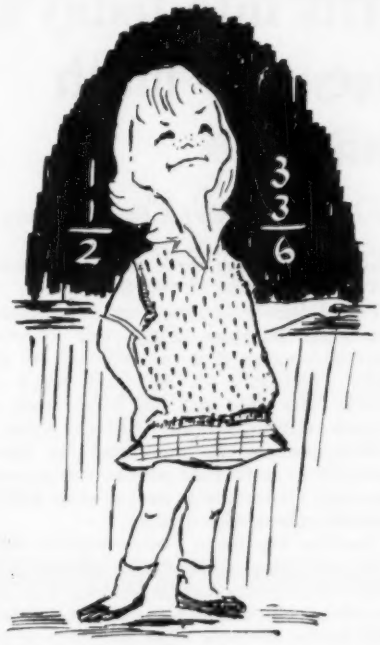
"MADISON PROJECT"

is one . . .

An NSF Academic Year Institute at Syracuse will enable teachers to study this, and a wide variety of other topics (from calculus to computers) during the year 1960-1961.

Participants will receive the usual NSF stipends, free tuition, annd travel allowance.

For information, write immediately to: Professor Robert B. Davis
Mathematics Department
Syracuse University
Syracuse 10, N.Y.



The answer is . . .

MATHEMATICS
IN DAILY USE, 3rd ed.
Hart • Schult • Irvin

A revised and completely up-to-date ninth grade text, designed to give pupils the practical, essential mathematics they will need to use every single day. Motivation is "built-in" because the problems in the text are so soundly practical and so obviously useful. Arithmetic processes are taught inductively and thoroughly; abundant practice material is provided; new concepts and skills are used immediately after they are introduced.

D. C. Heath and Company

Meets the challenge of today's world

The New Second Edition of the
Row-Peterson ALGEBRA PROGRAM
Algebra I and Algebra II . 1960 Copyright

Yes, here is an Algebra program that places heavy emphasis on showing the challenge in our modern world and in motivating students to meet that challenge through concentrated study and practice. It prepares them for tomorrow with a thorough, systematic, and intensive coverage of all the principles and processes of Algebra. Specially prepared test booklets consist of 32 tests for each grade. Teacher's Edition contains answers to all tests, plus detailed solutions of textbook problems.

Row, Peterson and Company

Evanston, Illinois — Elmsford, New York

How the slide rule fits into today's accelerated science and math program

ROBERT JONES, *Manager, Educational Sales Division, Frederick Post Company*

Increased government emphasis on secondary education poses this new challenge for mathematics teachers. Even before Sputnik I, our legislators in Washington took a long, hard look at the nation's secondary school curricula. The first result of their study: the National Defense Education Act of 1958. This Act in part calls for intensive emphasis on science and mathematics at the secondary school level. Much of the work which was formerly left for college must now be handled in high school. To meet these new requirements it's essential that pre-engineering or science majors learn the use of the slide rule *before* they get to college.

Unless they have guidance from you about what to look for in a slide rule, it's natural that your students would be attracted to the "bargain" slide rules that can be found almost anywhere selling for $.75 to $1.25. These rules have the basic scales, they *look* efficient to the unpracticed eye.

But are they *really* bargains? In the interest of better teaching and for the sake of your students, take a few minutes to evaluate these "bargain" slide rules for yourself. You'll undoubtedly find they have these weaknesses:

The graduations on "bargain" rules are printed or molded. Markings of this kind are often inaccurate, and almost always *temporary*. We've seen scores of rules on which markings fade after months of limited use. Is this a wise investment—at *any* price?

Another weakness of "bargain" rules is the basic material used. It will swell and contract under atmospheric changes. Once warped by these changes, the readings on these rules are often not dependable. The cursor is affected, too—it may bind or fit loosely and cause further inaccuracies.

At only slightly higher cost, we're sure you'll agree that the POST 10″ #1447 Student Slide Rule (Mannheim type) is far more worthwhile for the average student. It sells for $2.81 (classroom price) and offers sound value for every penny over and above the cost of "bargain" rules.

This slide rule is constructed of seasoned, laminated bamboo. POST has adopted bamboo because it is not affected by climatic conditions—it will not warp or shrink. The slide will not bind, stick or require artificial lubricants at any time.

The bamboo is laminated for extra strength as a further precaution against warping. Distortion is no problem with a bamboo rule of this quality.

Another feature of POST's Student Slide Rule is the engine-dividing of the graduations. Each graduation is precisely cut into the white celluloid face, making it a *permanent part of the rule.* The accuracy of the cuts is assured by modern machine controls.

The POST 1447 Student Slide Rule has the A, B, CI, D, and K scales on the face and the S, L, and T scales on the reverse side of the slide.

The cursor is framed in metal for durability, and a tension spring maintains the vertical position of the hairline. The hairline itself is etched in clear glass. The rule is furnished with a sturdy slip cap case and an easy-to-understand instruction booklet. An imitation leather case is available at slight additional cost. This rule serves the student dependably and accurately throughout his school years and on through his adult life.

Educators can help their students appreciably by advocating better slide rules (not necessarily expensive) for basic calculations. To prove our point, we'll be happy to supply you on a 90-day trial basis with one of these POST 10″ Student Slide Rules, along with a complimentary catalog. Please address your inquiry to *Educational Division, Frederick Post Company, 3650 N. Avondale Avenue, Chicago 18, Illinois.*

POST

FUNCTIONAL MATHEMATICS

by William A. Gager and others

. . . is integrated mathematics for the school years 7-12

. . . is for students of average and above average ability

. . . convinces students that mathematics is interesting and important, thus insuring retention

. . . is for those schools wishing to improve on the traditional approach to mathematics without the impossible extremes advocated by some groups

. . . has been bought in classroom quantities in forty-six states and by hundreds of schools.

For further information write to

EDUCATIONAL DEPARTMENT
CHARLES SCRIBNER'S SONS 597 Fifth Ave., New York 17, N.Y.

Three printings already required

INSIGHTS INTO MODERN MATHEMATICS

23rd Yearbook of the National Council of Teachers of Mathematics

What is meant by "modern mathematics"?

How much of it can be taught in the high school?

Has the point of view in mathematics been changed?

You cannot answer, or even understand, these questions without information.

CONTENTS

I. Introduction
II. The Concept of Number
III. Operating with Sets
IV. Deductive Methods in Mathematics
V. Algebra
VI. Geometric Vector Analysis and the Concept of Vector Space
VII. Limits
VIII. Functions
IX. Origins and Development of Concepts of Geometry
X. Point Set Topology
XI. The Theory of Probability
XII. Computing Machines and Automatic Decisions
XIII. Implications for the Mathematics Curriculum

$5.75 $4.75 to members of the Council
Postpaid if you send remittance with order.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W. Washington 6, D.C.

HOW TO—

NO. 1

How to Use Your Bulletin Board

Donovan A. Johnson
and Clarence E. Olander

Make your bulletin board a genuine teaching aid. Discusses purposes, appropriate topics, supplies needed, techniques, and "tricks of the trade."

12 pages. 50¢ each.

NO. 2

How to Develop a Teaching Guide in Mathematics

Mildred Keiffer and Anna Marie Evans

Discusses steps in the development of a guide, principles involved, content, and use. Contains an annotated bibliography of reference materials.

10 pages. 40¢ each.

NO. 3

How to Use Field Trips in Mathematics

Donovan A. Johnson, Dirk Ten Brinke, and Lauren G. Woodby

Discusses types of field trips, purposes, where to go, how to plan, follow-up activities, and the like. Gives examples of actual field trips.

8 pages. 35¢ each.

NO. 4

How to Use Films and Filmstrips in Mathematics Classes

Robert Vollmar and Philip Peak

Discusses the selection and use of films and filmstrips, including attitudes, objectives, preview and selection, procedures for effective use, and evaluation.

14 pages. 50¢ each.

NO. 5

How to Use Your Library in Mathematics

Allene Archer

Discusses purposes, outcomes, reference materials, topics, and projects. Information on historical reports, things to make, mathematicians, quotations.

6 pages. 40¢ each.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W.
Washington 6, D.C.

You owe it
to your students
to investigate the

MATHEMATICS STUDENT JOURNAL

A quarterly publication of the National Council of Teachers of Mathematics

For students from Grades 7 thru 12.

Redesigned in format, doubled in size, printed in two colors, expanded in range and content.

Contains material for enrichment, recreation, and instruction.

Features challenging problems and projects.

Circulation more than doubled during last year.

Two issues each semester, in November, January, March, and May.

Note these low prices:

Sold only in bundles of 5 copies or more. Price computed at single-copy rates of 30¢ per year, 20¢ per semester, making the minimum order only $1.50 per year or $1.00 per semester.

Order now for next semester

Please send remittance with your order.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N. W.
Washington 6, D. C.

Binders for the MATHEMATICS TEACHER

•

This handsome, durable, magazine binder has been restocked, due to wide demand. Designed to hold eight issues (one volume) either temporarily or permanently. Improved mechanism. Dark green cover with words "Mathematics Teacher" stamped in gold on cover and backbone. Issues may be inserted or removed separately. $2.50 each.

•

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W.
WASHINGTON 6, D.C.

DO YOU DREAD BLACKBOARD WORK?

New

TRY THE EASY, DUSTLESS WAY OF BLACKBOARD WRITING

NEW HAND-GIENIC, the automatic pencil that uses any standard chalk, ends forever messy chalk dust on your hands and clothes. No more recoiling from fingernails scratching on board, screeching or crumbling chalk. Scientifically balanced, fits hand like a fountain pen . . . chalk writing becomes a smooth pleasure. At a push of a button chalk ejects . . . retracts for carrying in pocket or purse. It's the "natural" gift for a fellow teacher, too!

STOPS CHALK WASTE—CHECKS ALLERGY Because HAND-GIENIC holds chalk as short as ¼" and prevents breakage, it allows the use of 95% of the chalk length in comparison with only 45% actually used without it. Hand never touches chalk during use, never gets dried up or infected from allergy.

STURDY METAL CONSTRUCTION for long, reliable service. 1-YR. WRITTEN GUARANTEE. Jewel-like 22K gold plated cap, onyx-black barrel. Distinctive to use, thoughtful to give. FREE TRIAL OFFER. Try it at our risk: Send $2 for one (or only $5 for set of 3). Postage free—no COD's. Enjoy HAND-GIENIC for 10 days, show it to other teachers. If not delighted, return for full refund. Ask for quantity discounts and Teacher-Representative plan. It's not sold in stores. ORDER TODAY.

HAND-GIENIC, Dept. 249, 2384 West Flagler, Miami 35, Fla.

MATHEMATICAL TEACHING MODELS

For every course where geometry is used, taught or applied, there is a model available for use. Theorems can be illustrated in seconds, and the impact of an exact picture remains fixed in the student's mind. These models are suitable for courses in:

- PLANE GEOMETRY
- SOLID GEOMETRY
- DESCRIPTIVE GEOMETRY
- MECHANICS
- MECHANICAL DRAWING
- INDUSTRIAL DESIGN

THEY HAVE THESE OUTSTANDING ADVANTAGES:

- CLEAR PLASTIC
- LIGHT WEIGHT
- CONVENIENT SIZE
- MODERATE COST
- SINGLE UNITS OR SETS

The clear plastic models permit the instructor to show the student where diagonals intersect INSIDE the bodies. The moving models in mechanics show the student what happens when gears mesh; the block type models permit the building of figures; the sets show the sections made by the intersections of cones and other solid figures by planes and by each other. Hundreds of models are available for almost every conceivable application.

MODEL 132: Plane section of a right circular cylinder (ellipse).

SPECIAL DISCOUNT SCHEDULE FOR SCHOOLS, COLLEGES AND UNIVERSITIES

WRITE FOR COMPLETE CATALOG

Physics Research LABORATORIES *House of Fine Instruments*

P.O. Box #555 Hempstead, L. I., N. Y.

MATHEMATICS LEARNING AND TEACHING AIDS

● To promote interest in Mathematical outside activities on the part of pupils, we are including an eight page section on Mathematics—on the importance of Mathematics to individuals and presenting learning aids—in our catalog which has a circulation of over a half a million so that the importance of Mathematics is brought to the attention of many people.

Some of the learning and teaching aids we will offer are shown here. We expect to be a regular advertiser in your magazine. Watch our advertisements for new items. We will greatly appreciate it if you as a teacher will recommend us to pupils interested in learning aids, books, etc.

D-STIX CONSTRUCTION KITS

You can increase interest in geometry and teach it better by using D-Stix. Solid geometry is fun for pupils when you show them actual visualizations of regular polyhedrons, geometric figures from triangles and cubes through such multiple sided figures as icosahedrons, dodecahedrons, etc. 230 pieces, 5, 6 and 8 sleeve connectors, 2", 3", 4", 5", 6" and 8" colored D-Stix.

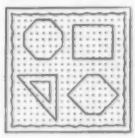
Stock No. 70,209-DH .....$3.00 Postpaid

370 pieces, including 5, 6 and 8 sleeve connectors, 2", 3", 4", 5", 6", 8", 10" and 12" D-Stix in colors.

Stock No. 70,210-DH .....$5.00 Postpaid

452 pieces, includes all items in 70,210 above, plus long unpainted D-Stix for use in making your own designs.

Stock No. 70,211-DH .....$7.00 Postpaid



RUB-R-ART

The use of this economical teaching aid in your classroom will increase your student's understandings of the perimeter and area of plane figures. This aid is 10" x 10" plastic peg board on which geometric representations can be made with rubber bands.

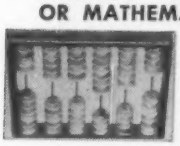
Stock No. 60,089-DH .....$1.00 Postpaid

WOODEN SOLIDS PUZZLES

Our sphere, cube, cylinder and octagonal prism wooden puzzles can be a big help in enriching your teaching of the volumes and lateral areas of solid figures. They are about 2" high. There are 13 puzzles in a set, including animal figures, etc. Take time in your teaching to let your students try to re-assemble these solid puzzles.

Stock No. 70,205-DH .....$3.00 Postpaid

FOR YOUR CLASSROOM LIBRARY OR MATHEMATICS LABORATORY



ABACUS

Our Abacus is just the thing for your gifted students to use in their enrichment units or for math clubs. It is our own design and is 9⅜" x 7⅜". It is made of a beautiful walnut wood, with 6 rows of 10 counters. Complete instructions are included with each abacus.

Stock No. 70,201-DH .....$4.95 Postpaid

ABACUS KIT—MAKE YOUR OWN!

Making your own Abacus is a wonderful project for any math class, or math club, or as an enrichment unit. Our kit gives you 60 counters, directions for making your own Abacus and directions for using our Abacus.—Makes one Abacus.

Stock No. 60,068-DH .....$ 1.30 Postpaid
Stock No. 70,226-DH .....$17.50 Postpaid
Gives you 1,000 counters and one set of directions. Makes 16 Abacuses.
Stock No. 50,234-DH .....$6.90 Postpaid
Gives you 100 brass rods for making Abacuses. Makes 16 Abacuses.
INSTRUCTION BOOKLET
Stock No. 9080-DH .....$ .25 ea. Postpaid
25 ..... 2.00 Postpaid
100 ..... 6.00 Postpaid

SLIDE RULE

We sell a bargain 10" plastic slide rule, a $7.00 value, for $3.00. These are perfect for math clubs, for teacher use, or for students wanting to calculate more quickly and accurately. A 14-page instruction booklet is given free with each rule.

Stock No. 30,288-DH .....$3.00 Postpaid

NEWEST TEACHING AID: TRIG AND CALCULUS CARDS

Our newest items that we are offering are 4 decks of Trig and Calculus Cards. These cards are a great asset to any math laboratory or math classroom. Now that more and more advanced math is being taught in high schools throughout the country, these are just the math teaching and learning aid that can be used to clinch the learning of trig identities or calculus formulas. Each deck contains 52 playing cards, plus instructions, and is used to play a game similar to Solitaire. Our decks include Differential Calculus, Integral Calculus, Applied Calculus, and Fundamental Identities from Trig.

Stock No. 40,310-DH—Applied Calculus .....$1.25 Postpaid
Stock No. 40,311-DH—Fundamental Identities ... 1.25 Postpaid
Stock No. 40,312-DH—Integral Calculus .....1.25 Postpaid
Stock No. 40,313-DH—Differential Calculus ..... 1.25 Postpaid
Stock No. 40,314-DH—Set of all four ..... 4.00 Postpaid

RADIAN PROTRACTOR

This excellent teaching and learning aid eliminates the need to learn long tables of equivalences or buy expensive books of tables for translating degrees to radian. Saves time in computing linear and angular velocity, area of circular segments, etc.

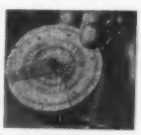
Stock No. 50,265-DH—Pkg. of 25 .....$ 3.75 Postpaid
Stock No. 50,266-DH—Pkg. of 100 .....$12.50 Postpaid

NEW, LOW-COST LAB PROJECTOR SHOWS EXPERIMENTS ON SCREEN!

New way to teach chemistry, biology. Project on-the-spot experiments, on screen or wall, with magnification, actually as they progress. Important phases, reactions may be observed by student group in revealing size—perfect vehicle for clear-cut instruction. Projector comes with a 3-element 80mm focal length f/3.5 anastigmat lens and a fast 28mm focal length, 4-element f/1.2 lens for microslide projection use. Also you get Prism Erecting system; special elevated slide and specimen projection stage; standard 35mm, 2" x 2" slide carrier; 35mm strip film holder. Additional accessories available—water cooled stage; polarizing filters, petri dishes; miniature test tubes and holders; gas absorption apparatus, electrolytic cells and many others.

Stock No. 70,230-DH .....$45.00 Postpaid

NEW! CIRCULAR SLIDE RULE!



POCKET SIZE—FAST—EASY TO USE!

Be a Math Whiz! New Circular Slide Rule multiplies, divides, figures fractions, percentages, squares, cubes, roots, proportions, circumferences, areas, retail prices, fuel consumption. Eliminates the confusions of ordinary slide rules. Priced far lower. Faster, easier to learn and use. Constructed of 2 aluminum discs with plastic indicator. 3½" diameter. Directions included.

Stock No. 30,336-DH .....$4.95 Postpaid

FREE CATALOG—DH

128 Pages! Over 1000 Bargains

America's No. 1 source of supply for low-cost Math and Science Teaching Aids, for experimenters, hobbyists. Complete line of Astronomical Telescope parts and assembled Telescopes. Also huge selection of lenses, prisms, war surplus optical instruments, parts and accessories. Telescopes, microscopes, satellite scopes, binoculars, infrared snipperscopes, etc.

Request Catalog—DH

ORDER BY STOCK NUMBER . . . SEND CHECK OR MONEY ORDER . . . SATISFACTION GUARANTEED!

EDMUND SCIENTIFIC CO. BARRINGTON, NEW JERSEY

Official U. S. Navy Photo

"Accurate volume calculations for this sphere-shaped satellite are obtained with the formula . . . $V = 4/3 \pi r^3$."

Discovering SOLIDS

A Series of Three Films
Applying Mathematics Principles to Space Perception

I. "VOLUMES OF CUBES, PRISMS, AND CYLINDERS"

II. "VOLUMES OF PYRAMIDS, CONES, AND SPHERES"

III. "SURFACE AREAS OF SOLIDS"

ART ANIMATION and MODEL DEMONSTRATION help develop formulas for finding volumes and areas of solids. LIVE FOOTAGE shows the use of these formulas in practical situations. Carefully produced under the supervision of DR. E. H. C. HILDEBRANDT of Northwestern University's Department of Mathematics, these films meet the demands of the re-vitalized mathematics curriculum.

PREVIEW PRINTS AVAILABLE

Junior High—High School
18 minutes
Color . . . . . $150.00 each
B & W . . . . . 75.00 each

Delta
FILM PRODUCTIONS, INC.
(DISTRIBUTION OFFICE)
1821 University Ave. St. Paul 4, Minnesota

THE GROWTH OF MATHEMATICAL IDEAS, GRADES K–12

24th Yearbook of the
National Council of Teachers of Mathematics

Attempts to suggest how basic and sound mathematical ideas, whether *modern* or traditional, can be made continuing themes in the development of mathematical understandings.

Defines and illustrates some classroom procedures which are important at all levels of instruction.

Discusses and illustrates *mathematical modes of thought.*

Gives suggestions to assist teachers and supervisors in applying the ideas of the book in their own situations.

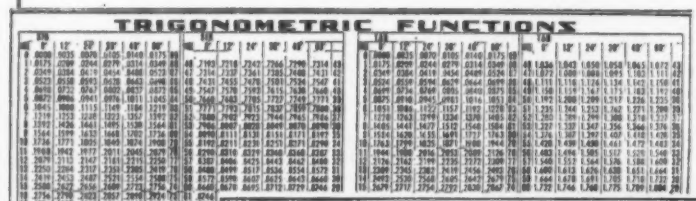
517 pp. $5.00 ($4.00 to members of the Council)

Postpaid if you send remittance with order.

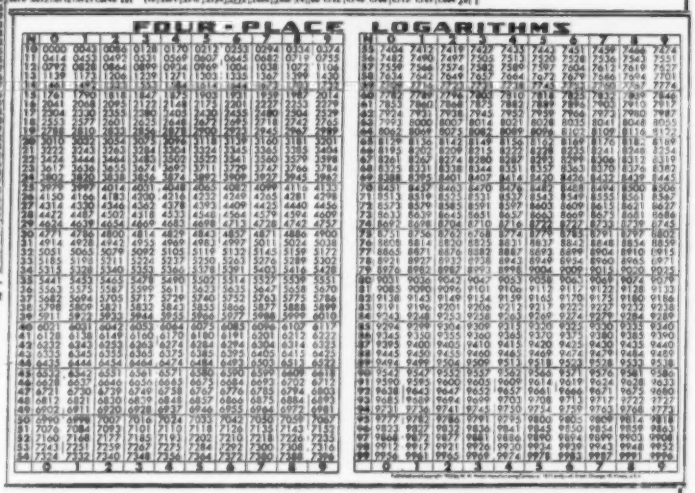
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
1201 Sixteenth Street, N.W. Washington 6, D.C.

The Welch Chart of

Logarithms and Trigonometric Functions
A reference for classroom teaching and for use in closed-book examinations



Large!
76 × 52
inches



Printed in red and black

No. 7550

- 4-place tables
- Excellent for teaching interpolation
- Trig functions are to every 12 minutes (0.2 of a degree)

The chart is reversible—logarithms are on one side, trigonometric functions on the other.

No. 7550. CHART, complete with reverse mechanism. Each, $15.00

This is but one of many moderate teaching aids for mathematics described in The Mathematics Instruments and Supplies Catalog. Write for your free copy.

W. M. WELCH SCIENTIFIC COMPANY
DIVISION OF W. M. WELCH MANUFACTURING COMPANY

1515 Sedgwick Street, Chicago 10, Illinois, U.S.A.